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1 Matrix Lie groups

Let $\mathrm{GL}(n)$ be the general linear group, i.e. the group of invertible $n \times n$ matrices under matrix multiplication. A closed subgroup of $\mathrm{GL}(n)$ is called a matrix Lie group. Examples of matrix Lie groups include:

$\mathrm{GL}(n)$	general linear group
$\mathrm{SL}(n)$	special linear group
$\mathrm{O}(n)$	general orthogonal group
$\mathrm{SO}(n)$	special orthogonal group
$\mathrm{Z}(n) \cong \mathbb{C}_{\neq 0}$	group of dilations
$\mathrm{SZ}(n)$	group of nth roots of unity
$\mathrm{CO}(n) = \mathrm{O}(n) \times \mathrm{Z}(n)$	conformal group
$\mathrm{T}(n) \cong \mathbb{C}^n$	group of translations
$\mathrm{S}(n) = \mathrm{CO}(n) \ltimes \mathrm{T}(n)$	similarities group
$\mathrm{Aff}(n) = \mathrm{GL}(n) \ltimes \mathrm{T}(n)$	general affine group
$\mathrm{SAff}(n) = \mathrm{SL}(n) \ltimes \mathrm{T}(n)$	special affine group
$\mathrm{E}(n) = \mathrm{O}(n) \ltimes \mathrm{T}(n)$	general euclidean group
$\mathrm{SE}(n) = \mathrm{SO}(n) \ltimes \mathrm{T}(n)$	special euclidean group
$\mathrm{PGL}(n) = \mathrm{GL}(n)/\mathrm{Z}(n)$	projective general linear group
$\mathrm{PSL}(n) = \mathrm{SL}(n)/\mathrm{SZ}(n)$	projective special linear group

The group we are most interested in within the context of computer vision is $\mathrm{PGL}(n)$, also called the group of homographies. Its dimension is

$$\dim \mathrm{PGL}(n) = \dim \mathrm{GL}(n)/\mathrm{Z}(n) = \dim \mathrm{GL}(n) - \dim \mathrm{Z}(n) = n^2 - 1$$

2 Matrix Lie algebras

Here are some examples of Lie groups and their corresponding Lie algebras:

Lie group	Lie algebra
general linear group $\mathrm{GL}(n)$ invertible $n \times n$ matrices	general linear algebra $\mathrm{gl}(n)$ all $n \times n$ matrices
orthogonal group $\mathrm{O}(n)$ orthogonal $n \times n$ matrices	orthogonal algebra $\mathrm{o}(n)$ skew-symmetric $n \times n$ matrices

Let G be a Lie group and $\mathfrak{g}$ be its Lie algebra. The exponential $\exp : \mathfrak{g} \to G$ maps elements of $\mathfrak{g}$ to elements of G. Assume G is a matrix Lie group. Then $\mathfrak{g}$ forms the tangent space to G at the identity, and

$$\exp A = \sum_{k \in \mathbb{N}} \frac{A^k}{k!}$$

For example, the matrix exponential of a skew-symmetric matrix is a rotation matrix:

$$\exp \theta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

The Frobenius inner product $\langle \cdot, \cdot \rangle_{\mathrm{F}} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is defined as

$$\langle A, B \rangle_{\mathrm{F}} = \operatorname{tr}(A^* B)$$

where A^* is the conjugate transpose of A and tr is the trace. The Frobenius inner product induces the Frobenius norm $\| \cdot \|_{\mathrm{F}} : \mathfrak{g} \to \mathbb{R}_{\geq 0}$

$$\|A\|_{\mathrm{F}} = \sqrt{\langle A, A \rangle_{\mathrm{F}}}$$

The Riemannian norm $\|\cdot\|_{\mathrm{R}} : G \to \mathbb{R}_{\geq 0}$ of a Lie group element is the infimum of the Frobenius norms of the Lie algebra elements that generate it:

$$\|A\|_{\mathrm{R}} = \inf\{\|B\|_{\mathrm{F}} : \exp B = A\}$$

The Riemannian norm induces a Riemannian metric $d_{\mathrm{R}} : G \times G \to \mathbb{R}_{\geq 0}$

$$d_{\mathrm{R}}(A, B) = \|AB^{-1}\|_{\mathrm{R}}$$

This metric is right-invariant:

$$\begin{aligned} d_{\mathrm{R}}(AC, BC) &= \|AC(BC)^{-1}\|_{\mathrm{R}} \\ &= \|ACC^{-1}B^{-1}\|_{\mathrm{R}} \\ &= \|AB^{-1}\|_{\mathrm{R}} \\ &= d_{\mathrm{R}}(A, B) \end{aligned}$$

2.1 Examples

In homogeneous coordinates, a translation is defined as

$$T(x, y) = \begin{bmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}$$

A rotation is defined as

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A scaling is defined as

$$S(a,b) = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A composition of all three is therefore

$$T(x,y)R(\theta)S(a,b) = \begin{bmatrix} a\cos\theta & -b\sin\theta & x \\ a\sin\theta & b\cos\theta & y \\ 0 & 0 & 1 \end{bmatrix}$$

Now notice that

$$\begin{aligned} \|T(x,y)\|_{\mathrm{R}} &= \sqrt{|x|^2 + |y|^2} \\ \|R(\theta)\|_{\mathrm{R}} &= \sqrt{2}|\theta| \\ \|S(a,b)\|_{\mathrm{R}} &= \sqrt{|\log a|^2 + |\log b|^2} \end{aligned}$$

The norm of each transformation agrees with our intuition about its “size”. In particular, the identity transformation is the “smallest” transformation:

$$\|T(0,0)\|_{\mathrm{R}} = \|R(0)\|_{\mathrm{R}} = \|S(1,1)\|_{\mathrm{R}} = \|I\|_{\mathrm{R}} = 0$$

3 The diffeomorphism group

Let $\mathcal{M}$ be a manifold and $p \in \mathcal{M}$. The tangent space at p, denoted by $\mathrm{T}_p\mathcal{M}$, is the set of all tangent vectors at p. A vector field on $\mathcal{M}$ is an assignment of a tangent vector to each point in $\mathcal{M}$. Let

$$\mathrm{Vec}^r(\mathcal{M}) \subset \prod_{p \in \mathcal{M}} \mathrm{T}_p\mathcal{M}$$

be the set of vector fields on $\mathcal{M}$ that are r-times continuously differentiable. A C^r diffeomorphism is an r-times continuously differentiable bijection between manifolds whose inverse is also r-times continuously differentiable. A C^0 diffeomorphism is a homeomorphism. A C^∞ diffeomorphism is *smooth*.

Let $\mathrm{Diff}^r(\mathcal{M})$ be the set of C^r diffeomorphisms from $\mathcal{M}$ to itself, which forms a group. Then the exponential map

$$\exp : \mathrm{Vec}^r(\mathcal{M}) \to \mathrm{Diff}^r(\mathcal{M})$$

is defined by

$$\exp f = \phi(1)$$

where $\phi : \mathbb{R} \to \mathrm{Diff}^r(\mathcal{M})$ is defined by

$$\begin{aligned} \phi(0) &= \mathrm{id}_{\mathcal{M}} \\ \dot{\phi}(t) &= f \circ \phi(t) \end{aligned}$$

In particular, if f is the zero vector field, then $\exp f = \mathrm{id}_{\mathcal{M}}$. Therefore the exponential of a vector field is the result of *following the flow* of the vector field for a unit amount of time, for every point on the manifold.

For example, suppose $\mathcal{M} = \mathbb{R}^n$. Then, for all $b \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$,

$$\begin{aligned} f(p) &= b & &\implies & (\exp f)(p) &= p + b \\ f(p) &= Ap & &\implies & (\exp f)(p) &= (\exp A)p \end{aligned}$$

Like before, the Riemannian norm of a diffeomorphism is the infimum of the Frobenius norms of the vector fields that generate it:

$$\|g\|_{\mathrm{R}} = \inf\{\|f\|_{\mathrm{F}} : \exp f = g\}$$

where

$$\|f\|_{\mathrm{F}} = \sqrt{\langle f, f \rangle_{\mathrm{F}}} = \sqrt{\int_{\mathcal{M}} \langle f, f \rangle \, \mathrm{d}V}$$

Let $\psi : \mathbb{R} \to (\mathcal{M} \to \mathcal{M})$ be the geodesic flow defined by

$$\begin{aligned} \psi(0) &= \mathrm{id}_{\mathcal{M}} \\ \dot{\psi}(0) &= f \\ \nabla_{\dot{\psi}} \dot{\psi} &= 0 \end{aligned}$$

Then $\phi(t) \approx \psi(t)$ for small t because $\phi(0) = \mathrm{id}_{\mathcal{M}} = \psi(0)$ and

$$\begin{aligned} \dot{\phi}(0) &= f \circ \phi(0) \\ &= f \circ \mathrm{id}_{\mathcal{M}} \\ &= f \\ &= \dot{\psi}(0) \end{aligned}$$

That is, the exponential of a vector field can be approximated by the geodesic flow whose initial velocity is that vector field. Thus we can approximate the set of diffeomorphisms with *displacement* fields rather than *velocity* fields, which is easier in the context of $\mathbb{R}^n$ because we only need to add a displacement vector to any given point, instead of numerically integrating the flow trajectory starting from that point.