Playing with differential operators

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1 General properties

The derivative of a function at a point can be defined by

$$\lim_{y\to x} \frac{f(y)-f(x)}{y-x} \tag{1}$$

Letting $y = x + h$ we obtain

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \tag{2}$$

The differential operator ∂_x is linear:

$$\partial_x(u+v) = \partial_x u + \partial_x v \tag{3}$$
$$\partial_x cu = c\partial_x u \tag{4}$$

for any constant c. It satisfies the product rule

$$\partial_x uv = (\partial_x u)v + u(\partial_x v) \tag{5}$$

The iterated product rule is given by the formula

$$\partial_x^n uv = \sum_{k=0}^{n} \binom{n}{k} (\partial_x^{n-k} u)(\partial_x^k v) \tag{6}$$

2 Solving homogeneous equations

Suppose we have a differential equation of the form

$$\partial_x y = y \tag{7}$$

The solution is given by $y = C\mathrm{e}^x$ where C is any constant. In general,

$$\partial_x y = \lambda y \tag{8}$$

has the solution $y = C\mathrm{e}^{\lambda x}$ since

$$\partial_x \mathrm{e}^{\lambda x} = \lambda \mathrm{e}^{\lambda x} \tag{9}$$

Hence λ is an eigenvalue of the differential operator with eigenfunction $\mathrm{e}^{\lambda x}$. Stated differently, $y = C\mathrm{e}^{\lambda x}$ is the solution to an equation of the form

$$(\partial_x - \lambda)y = 0 \tag{10}$$

where we "factor out" y from the expression. Now consider a more general differential equation of the form

$$a\partial_x y + by = 0 \tag{11}$$

This can be solved by "factoring" y from the expression once again:

$$\begin{aligned} 0 &= (a\partial_x + b)y \\ &= a(\partial_x + a^{-1}b)y \end{aligned} \tag{12}$$

Therefore, using the zero-product property of multiplication,

$$0 = (\partial_x + a^{-1}b)y \tag{13}$$

We already saw that the solution to an equation of this form is $y = C\mathrm{e}^{\lambda x}$, where in this case $\lambda = -a^{-1}b$. Hence the solution is $y = C\mathrm{e}^{-a^{-1}bx}$. Consider next the second-order differential equation:

$$a\partial_x^2 y + b\partial_x y + cy = 0 \tag{14}$$

Again, we factor out y:

$$0 = (a\partial_x^2 + b\partial_x + c)y \tag{15}$$

We can treat $a\partial_x^2 + b\partial_x + c$ as a polynomial in ∂_x and find its zeros, which are given by the quadratic formula:

$$\begin{aligned} r_1 &= \frac{-b + \sqrt{b^2 - 4a}}{2a} \\ r_2 &= \frac{-b - \sqrt{b^2 - 4a}}{2a} \end{aligned} \tag{16}$$

to express it as a product of linear factors:

$$a\partial_x^2 + b\partial_x + c = a(\partial_x - r_1)(\partial_x - r_2) \tag{17}$$

Hence our problem is reduced to

$$0 = a(\partial_x - r_1)(\partial_x - r_2)y \tag{18}$$

which, by the zero-product property, reduces to solving

$$\begin{aligned} 0 &= (\partial_x - r_1)y \\ 0 &= (\partial_x - r_2)y \end{aligned} \tag{19}$$

Hence the two solutions are $y = \mathrm{e}^{r_1 x}$ and $y = \mathrm{e}^{r_2 x}$. Because this is a homogeneous equation and the differential operator is linear, any linear combination of solutions is also a solution. Therefore, the general solution is given by

$$y = C_1 \mathrm{e}^{r_1 x} + C_2 \mathrm{e}^{r_2 x} \tag{20}$$

provided $r_1 \neq r_2$. Generalizing these results, we can find the general solution of an arbitrary differential equation of the form

$$\sum_{k=0}^{n} a_n \partial_x^k y = \left(\sum_{k=0}^{n} a_k \partial_x^k \right) y = P(\partial_x) y \tag{21}$$

where P is a polynomial by finding the roots of P and expressing the solution as a linear combination of the solutions for each root:

$$y = \sum_{k=0}^{n} C_k \mathrm{e}^{r_k x} \tag{22}$$

provided there are no repeated roots.

3 Solving inhomogeneous equations

Consider, on the other hand, an inhomogeneous differential equation:

$$y - \partial_x y = Q(x) \tag{23}$$

Factoring y yields

$$(1 - \partial_x) y = Q(x) \tag{24}$$

which seems to lead us nowhere... or does it? Let's take a leap:

$$y = (1 - \partial_x)^{-1} Q(x) \tag{25}$$

Recall that the sum of an infinite geometric series is given by

$$\frac{1}{1 - x} = \sum_{k=0}^{\infty} x^k \tag{26}$$

for $|x| < 1$. Substituting this into our differential equation yields

$$
\begin{aligned}
y &= \left(\sum_{k=0}^{\infty} \partial_x^k\right) Q(x) \\
&= \sum_{k=0}^{\infty} \partial_x^k Q(x)
\end{aligned}
\tag{27}
$$

We found the solution! The solution is given by the sum of the derivatives of Q. If Q is a finite polynomial, the sum will have a finite number of terms as well, since the kth derivative of a nth degree polynomial is zero for $k > n$. Still don't believe it? Try a few different $Q(x)$ for yourself. In general,

$$
\frac{1}{x-r} = -\frac{1}{r}\frac{1}{1-xr^{-1}} = -\frac{1}{r}\sum_{k=0}^{\infty}(xr^{-1})^k = -\sum_{k=0}^{\infty} x^k r^{-(k+1)} \tag{28}
$$

Hence we know how to solve a differential equation of the form

$$
P(\partial_x)y = Q(x) \tag{29}
$$

since we can write P as a product of $x - r$ factors, where r are the roots of P, and then use the technique we just derived. For example, consider

$$
y - \partial_x^2 y = 2 + 3x^2 \tag{30}
$$

We solve it as follows:

$$
\begin{aligned}
(1-\partial_x^2)y &= 2 + 3x^2 \\
(1+\partial_x)(1-\partial_x)y &= 2 + 3x^2 \\
y &= (1-\partial_x)^{-1}(1+\partial_x)^{-1}(2+3x^2) \\
&= \left(\sum_{k=0}^{\infty} \partial_x^k\right)\left(\sum_{k=0}^{\infty}(-1)^k \partial_x^k\right)(2+3x^2) \\
&= \left(\sum_{k=0}^{\infty} \partial_x^k\right)\sum_{k=0}^{\infty}(-1)^k \partial_x^k(2+3x^2)
\end{aligned}
\tag{31}
$$

Because $2+3x^2$ is a second degree polynomial, we need only concern ourselves with $k \leq 2$ for the sum on the right:

$$
\begin{aligned}
y &= \left(\sum_{k=0}^{\infty} \partial_x^k\right) \sum_{k=0}^{2} (-1)^k \partial_x^k (2+3x^2) \\
&= \left(\sum_{k=0}^{\infty} \partial_x^k\right) \left((-1)^0 \partial_x^0 (2+3x^2) + (-1)^1 \partial_x^1 (2+3x^2) + (-1)^2 \partial_x^2 (2+3x^2)\right) \\
&= \left(\sum_{k=0}^{\infty} \partial_x^k\right) (2+3x^2-6x+6) \\
&= \left(\sum_{k=0}^{\infty} \partial_x^k\right) (3x^2-6x+8)
\end{aligned}
\tag{32}
$$

Again, we we need only concern ourselves with $k \leq 2$:

$$
\begin{aligned}
y &= \left(\sum_{k=0}^{\infty} \partial_x^k\right) (3x^2-6x+8) \\
&= \partial_x^0 (3x^2-6x+8) + \partial_x^1 (3x^2-6x+8) + \partial_x^2 (3x^2-6x+8) \\
&= 3x^2-6x+8+6x-6+6 \\
&= 3x^2+8
\end{aligned}
\tag{33}
$$

which indeed satisfies the inhomogeneous differential equation. The remaining components of the general solution are provided by the general solution to the corresponding homogeneous equation:

$$
y - \partial_x^2 y = 0 \tag{34}
$$

namely $y = C_1 \mathrm{e}^x + C_2 \mathrm{e}^{-x}$. Why? Because if y_I is a solution to the inhomogeneous equation and y_H is a solution to the homogeneous equation then

$$
\begin{aligned}
P(\partial_x)(y_I + y_H) &= P(\partial_x) y_I + P(\partial_x) y_H \\
&= Q(x) + 0 \\
&= Q(x)
\end{aligned}
\tag{35}
$$

by the linearity of the differential operator. Therefore, the complete solution to the inhomogeneous equation is given by

$$
\begin{aligned}
y &= y_I + y_H \\
&= 3x^2 + 8 + C_1 \mathrm{e}^x + C_2 \mathrm{e}^{-x}
\end{aligned}
\tag{36}
$$

4 The commutator

While the differential operator ∂_x commutes with any constant b, as in $\partial_x b = b\partial_x$, it does not commute with x. The commutator of ∂_x and x is nonzero:

$$[\partial_x, x] = \partial_x x - x\partial_x \neq 0 \tag{37}$$

In fact, it turns out that their commutator is 1. This can be shown through a simple application of the product rule:

$$\begin{aligned}
[\partial_x, x]f &= (\partial_x x - x\partial_x)f \\
&= \partial_x x f - x\partial_x f \\
&= f\partial_x x + x\partial_x f - x\partial_x f \\
&= f
\end{aligned} \tag{38}$$

Eliminating f from both sides yields

$$[\partial_x, x] = 1 \tag{39}$$

The fact that x and ∂_x do not commute is important in quantum mechanics, where it forms the basis of the uncertainty principle. The momentum operator is given there by $p = \mathrm{i}\hbar\partial_x$. Applying the same procedure yields

$$\begin{aligned}
[p, x]f &= [\mathrm{i}\hbar\partial_x, x]f \\
&= \mathrm{i}\hbar[\partial_x, x]f \\
&= \mathrm{i}\hbar
\end{aligned} \tag{40}$$

Therefore

$$[p, x] = \mathrm{i}\hbar \tag{41}$$

This is the canonical commutation relation. In general,

$$[p_i, x_j] = \mathrm{i}\hbar\delta_{ij} \tag{42}$$

where δ_{ij} is the Kronecker delta:

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{43}$$

If the components i and j of the position and momentum are different, they do commute and can be measured simultaneously.

Because x and ∂_x do not commute, we must be careful to avoid manipulations that rely on the assumption that they do.

5 Powers of the operator

Can we find a closed-form expression for $(x\partial_x)^n$? $x^n\partial_x^n$ is not the answer because ∂_x and x do not commute, as we showed in the previous section.

Let us try a few examples. Using the product rule

$$\partial_x ab = (\partial_x a)b + a(\partial_x b) \tag{44}$$

yields

$$\begin{aligned}
(x\partial_x)^1 &= x\partial_x \\
(x\partial_x)^2 &= x\partial_x + x^2\partial_x^2 \\
(x\partial_x)^3 &= x\partial_x + 3x^2\partial_x^2 + x^3\partial_x^3 \\
(x\partial_x)^4 &= x\partial_x + 7x^2\partial_x^2 + 6x^3\partial_x^3 + x^4\partial_x^4 \\
(x\partial_x)^5 &= x\partial_x + 15x^2\partial_x^2 + 25x^3\partial_x^3 + 10x^4\partial_x^4 + x^5\partial_x^5
\end{aligned} \tag{45}$$

This pattern is reminiscent of Pascal's triangle and the binomial expansion. Notice, for example, the triangular numbers $(1,\ 3,\ 6,\ 10,\ \dots)$ on the second diagonal. However, this pattern is non-symmetrical and the powers ascend in the same direction (as opposed to ascending in one term and descending in the other, as in the binomial expansion). It turns out these coefficients are given by the formula

$$(x\partial_x)^n = \sum_{k=1}^{n} \left\{ {n \atop k} \right\} x^k \partial_x^k \tag{46}$$

where $\left\{ {n \atop k} \right\}$ is a Stirling number of the second kind, or the number of ways to partition a set of n objects into k non-empty subsets. They satisfy the recurrence relation

$$\left\{ {n+1 \atop k} \right\} = \left\{ {n \atop k} \right\} k + \left\{ {n \atop k-1} \right\} \tag{47}$$

for $k > 0$ with initial conditions

$$\left\{ {0 \atop 0} \right\} = 1 \text{ and } \left\{ {n \atop 0} \right\} = \left\{ {0 \atop n} \right\} = 0 \tag{48}$$

for $n > 0$. We can prove the Stirling numbers do indeed yield the coefficients of $(x\partial_x)^n$ using mathematical induction.

Base case: Let $n = 1$. Then

$$\begin{aligned}
(x\partial_x)^1 &= x\partial_x \\
&= \left\{ {1 \atop 1} \right\} x^1 \partial_x^1
\end{aligned} \tag{49}$$

Therefore P_1 is true.
Induction step: Assume P_m is true.

$$(x\partial_x)^m = \sum_{k=1}^{m} \left\{ m \atop k \right\} x^k \partial_x^k \tag{50}$$

Consider P_{m+1}:

$$(x\partial_x)^{m+1} = x\partial_x (x\partial_x)^m \tag{51}$$

We use our induction hypothesis:

$$\begin{aligned}
(x\partial_x)^{m+1} &= x\partial_x \sum_{k=1}^{m} \left\{ m \atop k \right\} x^k \partial_x^k \\
&= x \sum_{k=1}^{m} \left\{ m \atop k \right\} \partial_x (x^k \partial_x^k) \\
&= x \sum_{k=1}^{m} \left\{ m \atop k \right\} ((\partial_x x^k)\partial_x^k + x^k(\partial_x \partial_x^k)) \\
&= x \sum_{k=1}^{m} \left\{ m \atop k \right\} (kx^{k-1}\partial_x^k + x^k \partial_x^{k+1}) \\
&= \sum_{k=1}^{m} \left\{ m \atop k \right\} (kx^k\partial_x^k + x^{k+1} \partial_x^{k+1})
\end{aligned} \tag{52}$$

Splitting into two sums and shifting the indices of the second sum yields

$$\begin{aligned}
(x\partial_x)^{m+1} &= \sum_{k=1}^{m} \left\{ m \atop k \right\} kx^k\partial_x^k + \sum_{k=1}^{m} \left\{ m \atop k \right\} x^{k+1}\partial_x^{k+1} \\
&= \sum_{k=1}^{m} \left\{ m \atop k \right\} kx^k\partial_x^k + \sum_{k=2}^{m+1} \left\{ m \atop k-1 \right\} x^k\partial_x^k
\end{aligned} \tag{53}$$

Since

$$\left\{ m \atop m+1 \right\} (m+1)x^{k+1}\partial_x^{k+1} = 0 \cdot (m+1)x^{k+1}\partial_x^{k+1} = 0 \tag{54}$$

we can extend the first sum to $k = m+1$:

$$(x\partial_x)^{m+1} = \sum_{k=1}^{m+1} \left\{ m \atop k \right\} kx^k\partial_x^k + \sum_{k=2}^{m+1} \left\{ m \atop k-1 \right\} x^k\partial_x^k \tag{55}$$

Again, since

$$\left\{ {m \atop 1-1} \right\} x^1 \partial_x^1 = \left\{ {m \atop 0} \right\} x\partial_x = 0 \cdot x\partial_x = 0 \tag{56}$$

we can extend the second sum to $k = 1$:

$$\begin{aligned}(x\partial_x)^{m+1} &= \sum_{k=1}^{m+1} \left\{ {m \atop k} \right\} k x^k \partial_x^k + \sum_{k=1}^{m+1} \left\{ {m \atop k-1} \right\} x^k \partial_x^k \\ &= \sum_{k=1}^{m+1} \left(\left\{ {m \atop k} \right\} + \left\{ {m \atop k-1} \right\} \right) x^k \partial_x^k\end{aligned} \tag{57}$$

Using our recurrence relation for the Stirling numbers, we obtain

$$(x\partial_x)^{m+1} = \sum_{k=1}^{m+1} \left\{ {m+1 \atop k} \right\} x^k \partial_x^k \tag{58}$$

Thus we have shown that if P_m is true then P_{m+1} is true. Earlier we proved that P_1 is true. Therefore, by induction, P_n is true for all $n \geq 1$.

6 The Lagrange shift operator

The Taylor series of a function f about a point $x = a$ is

$$\begin{aligned}f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= \sum_{n=0}^{\infty} \frac{(\partial_x^n f)(a)}{n!}(x-a)^n\end{aligned} \tag{59}$$

Expanding about $x + b$ yields

$$f(x+b) = \sum_{n=0}^{\infty} \frac{(\partial_x^n f)(a)}{n!}(x+b-a)^n \tag{60}$$

Letting $a = x$ yields

$$\begin{aligned}f(x+b) &= \sum_{n=0}^{\infty} \frac{(\partial_x^n f)(x)}{n!} b^n \\ &= \sum_{n=0}^{\infty} \frac{(b^n \partial_x^n f)(x)}{n!} \\ &= \sum_{n=0}^{\infty} \frac{((b\partial_x)^n f)}{n!}(x)\end{aligned} \tag{61}$$

Using the definition of the exponential

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{62}$$

We obtain

$$e^{b\partial_x} = \sum_{n=0}^{\infty} \frac{(b\partial_x)^n}{n!} \tag{63}$$

Therefore

$$f(x+b) = (e^{b\partial_x} f)(x) \tag{64}$$

Hence $T = e^{\partial_x}$ is an operator that shifts the argument of a function by 1, and T^b shifts it by b. In terms of the exponential map, $\exp : \mathfrak{g} \to G$, the differential operator ∂_x is the generator of "infinitesimal" translations.

7 The scaling operator

We can also find a closed form for an operator S such that $(Sf)(x) = f(xa)$:

$$\begin{aligned} f(xa) &= f(e^{\log xa}) \\ &= f(e^{\log x + \log a}) \\ &= f(e^{y + \log a}) \end{aligned} \tag{65}$$

where $y = \log x$. Let $g(z) = f(e^z)$:

$$f(xa) = g(y + \log a) \tag{66}$$

We can now use the shift operator we derived in the previous section:

$$g(y+b) = (e^{b\partial_y} g)(y) \tag{67}$$

Hence

$$\begin{aligned} f(xa) &= g(y + \log a) \\ &= (e^{(\log a)\partial_y} g)(y) \\ &= (a^{\partial_y} g)(y) \\ &= (a^{\partial_{\log x}} g)(y) \\ &= (a^{\partial_{\log x}} f)(e^{\log x}) \\ &= (a^{\partial_{\log x}} f)(x) \end{aligned} \tag{68}$$

Now since

$$\begin{aligned}\frac{\partial}{\partial \log x} &= \frac{\partial x}{\partial \log x}\frac{\partial}{\partial x} \\ &= \left(\frac{\partial \log x}{\partial x}\right)^{-1}\frac{\partial}{\partial x} \\ &= \left(\frac{1}{x}\right)^{-1}\frac{\partial}{\partial x} \\ &= x\frac{\partial}{\partial x}\end{aligned} \tag{69}$$

Then

$$f(xa) = (a^{x\partial_x} f)(x) \tag{70}$$

Therefore $S = a^{x\partial_x}$ is our scaling operator. $x\partial_x = \partial_{\log x}$ is thus the generator of "infinitesimal" dilations or scalings.

8 A general operator

Suppose we want an operator G such that $(Gf)(x) = f(g(x))$. Consider:

$$\begin{aligned}(\mathrm{e}^{\partial_{h(x)}})(x) &= (\mathrm{e}^{\partial_y} f)(x) \\ &= (\mathrm{e}^{\partial_y} f)(h^{-1}(h(x))) \\ &= (\mathrm{e}^{\partial_y} f)(h^{-1}(y))\end{aligned} \tag{71}$$

where $y = h(x)$. Let $j = f \circ h^{-1}$:

$$\begin{aligned}(\mathrm{e}^{\partial_{h(x)}})(x) &= (\mathrm{e}^{\partial_y}(f \circ h^{-1}))(y) \\ &= (\mathrm{e}^{\partial_y} j)(y) \\ &= j(y+1) \\ &= f(h^{-1}(y+1)) \\ &= f(h^{-1}(h(x)+1))\end{aligned} \tag{72}$$

Hence solving the functional equation

$$h^{-1}(h(x)+1) = g(x) \tag{73}$$

for $h(x)$ allows us to define our general operator $G = \mathrm{e}^{\partial_{h(x)}}$. For example, letting $g(x) = xa$, the function $h(x) = (\log a)^{-1} \log x$ is a solution:

$$h^{-1}(x) = a^x \tag{74}$$

$$
\begin{aligned}
h^{-1}(h(x)) &= a^{h(x)} \\
&= a^{(\log a)^{-1}\log x} \\
&= \mathrm{e}^{(\log a)(\log a)^{-1}\log x} \\
&= \mathrm{e}^{\log x} \\
&= x
\end{aligned} \tag{75}
$$

Therefore

$$
\begin{aligned}
h^{-1}(h(x)+1) &= \mathrm{e}^{(\log a)(h(x)+1)} \\
&= a^{h(x)+1} \\
&= a^{(\log a)^{-1}\log x+1} \\
&= a^{(\log a)^{-1}\log x} a \\
&= \mathrm{e}^{\log x} a \\
&= xa
\end{aligned} \tag{76}
$$

Hence our operator G takes the form

$$
\begin{aligned}
G &= \mathrm{e}^{\partial_{h(x)}} \\
&= \mathrm{e}^{\partial_{(\log a)^{-1}\log x}} \\
&= \mathrm{e}^{(\log a)\partial_{\log x}} \\
&= a^{x\partial_x}
\end{aligned} \tag{77}
$$

This is the scaling operator that we derived in the previous section. On the other hand, the case where $\partial_{h(x)} = x^n \partial_x$ or equivalently $h(x) = (1-n)^{-1}x^{1-n}$ gives us the basis for the Witt algebra, a Lie algebra. For example,

$$
(\mathrm{e}^{cx^2\partial_x} f)(x) = f\left(\frac{x}{1-cx}\right) \tag{78}
$$

9 The linear canonical transform

The linear canonical transform (LCT) is a family of integral transforms that generalizes many classical transforms, such as the Fourier, Laplace, and Weierstrass transforms. Formally speaking, LCT is the action of $\mathrm{SL}_2(\mathbb{R})$ (the special linear group of 2×2 real matrices with unit determinant) on the plane.

The Fourier transform, for example, corresponds to a rotation by 90°

$$
\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{79}
$$

The Laplace transform corresponds to an imaginary rotation by 90°:

$$\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \tag{80}$$

It can be shown that the linear canonical transformations can be expressed in the form $\mathrm{e}^{P_2(x,\partial_x)}$, where $P_2(x,\partial_x)$ is a complex polynomial of order ≤ 2 in x and ∂_x. For example, the Weierstrass transform can be formally expressed as

$$\mathcal{W} = \mathrm{e}^{\partial_x^2} \tag{81}$$

since

$$\mathrm{e}^{u^2} = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-yu} \mathrm{e}^{-y^2/4} \mathrm{d}y \tag{82}$$

and thus

$$\begin{aligned} (\mathrm{e}^{D^2} f)(x) &= \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-yD} \mathrm{e}^{-y^2/4} \mathrm{d}y \\ &= \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} f(x-y) \mathrm{e}^{-y^2/4} \mathrm{d}y \\ &= (\mathcal{W} f)(x) \end{aligned} \tag{83}$$

Similarly, the Fourier transform can be expressed as

$$\begin{aligned} \mathcal{F} &= (-i)^{\hat{N}} \\ &= (-i)^{\frac{1}{2}(x^2 - \partial_x^2 - 1)} \\ &= (\mathrm{e}^{-\frac{1}{2} i\pi})^{\frac{1}{2}(x^2 - \partial_x^2 - 1)} \\ &= \mathrm{e}^{\frac{i\pi}{4}(x^2 - \partial_x^2 - 1)} \end{aligned} \tag{84}$$

where $\hat{N}$ is the number operator in the quantum harmonic oscillator.

10 General function of a differential operator

In general, if $g(\partial_x)$ is a function g of the differential operator ∂_x, then

$$(g(\partial_x) f)(x) = \int_{-\infty}^{\infty} g(k) \hat{f}(k) \mathrm{e}^{2\pi i k x} \mathrm{d}k \tag{85}$$

where $\hat{f}$ is the Fourier transform of f. This is a consequence of the fact that the Fourier transform $\mathcal{F}$ satisfies $\mathcal{F}(\partial_x u) = x(\mathcal{F} u)$.

11 The Euler-Maclaurin formula

The sum of an infinite geometric series is

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \tag{86}$$

Replacing x with e^x yields

$$\frac{1}{1-\mathrm{e}^x} = \sum_{n=0}^{\infty} \mathrm{e}^{nx} \tag{87}$$

Multiplying both sides by x yields

$$\frac{x}{1-\mathrm{e}^x} = \sum_{n=0}^{\infty} \mathrm{e}^{nx} x \tag{88}$$

Now, the generating function for the Bernoulli numbers B_n is

$$\frac{x}{\mathrm{e}^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n \tag{89}$$

Hence

$$0 = \sum_{n=0}^{\infty} \mathrm{e}^{nx} x + \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n \tag{90}$$

Replace x with the differential operator ∂_x:

$$0 = \sum_{n=0}^{\infty} \mathrm{e}^{n\partial_x} \partial_x + \sum_{n=0}^{\infty} \frac{B_n}{n!} \partial_x^n \tag{91}$$

Replace $\mathrm{e}^{n\partial_x}$ with the shift operator T^n:

$$0 = \sum_{n=0}^{\infty} T^n \partial_x + \sum_{n=0}^{\infty} \frac{B_n}{n!} \partial_x^n \tag{92}$$

Applying this to a function f yields

$$0 = \sum_{n=0}^{\infty} (T^n \partial_x f)(x) + \sum_{n=0}^{\infty} \frac{B_n}{n!} (\partial_x^n f)(x) \tag{93}$$

Using $(T^n g)(x) = g(x+n)$ we obtain

$$
\begin{aligned}
0 &= \sum_{n=0}^{\infty}(\partial_x f)(x+n) + \sum_{n=0}^{\infty}\frac{B_n}{n!}(\partial_x^n f)(x) \\
&= \sum_{n=0}^{\infty}(\partial_x f)(x+n) + \frac{B_0}{0!}f(x) + \sum_{n=1}^{\infty}\frac{B_n}{n!}(\partial_x^n f)(x) \\
&= \sum_{n=0}^{\infty}(\partial_x f)(x+n) + f(x) + \sum_{n=0}^{\infty}\frac{B_{n+1}}{(n+1)!}(\partial_x^{n+1} f)(x)
\end{aligned} \tag{94}
$$

Letting $g = \partial_x f$ yields

$$
0 = \sum_{n=0}^{\infty} g(x+n) + \int_a^x g(t)\mathrm{d}t + \sum_{n=0}^{\infty}\frac{B_{n+1}}{(n+1)!}(\partial_x^n g)(x) \tag{95}
$$

which is a version of the Euler-Maclaurin formula. Note that it allows us to express the sum of a function evaluated at several points in terms of the integral and derivative of the function at a single point.

Now try to do the same for finite sums using the formula

$$
\sum_{n=0}^{k} x^n = \frac{1-x^{k+1}}{1-x} \tag{96}
$$

12 Playing fast and loose with operators

Let $\int$ denote $\int_0^x \mathrm{d}x$. Consider the integral equation

$$
f - \int f = 1 \tag{97}
$$

Factor out f:

$$
\left(1 - \int\right) f = 1 \tag{98}
$$

Multiply both sides by $\left(1-\int\right)^{-1}$ and use the geometric series formula:

$$
\begin{aligned}
f &= \left(1 - \int\right)^{-1} 1 \\
&= \sum_{n=0}^{\infty} \int^n 1
\end{aligned} \tag{99}
$$

We can interpret the $\int^n$ as iterated integrals:

$$
\begin{aligned}
f &= 1 + \int_0^x \mathrm{d}x\, 1 + \int_0^x \mathrm{d}x \int_0^x \mathrm{d}x'\, 1 + \dots \\
&= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \\
&= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\
&= \mathrm{e}^x
\end{aligned} \tag{100}
$$

This is indeed the solution to the original integral equation!

13 Integral of a product with a monomial

Suppose we want to find a closed form the for expression

$$\int x^n f(x) \tag{101}$$

This would be useful for an expression of the form

$$\int P(x) f(x) \tag{102}$$

where $P(x)$ is a polynomial of arbitrary degree m, since

$$
\begin{aligned}
\int P(x) f(x) &= \int \sum_{n=0}^{m} a_n x^n f(x) \\
&= \sum_{n=0}^{m} a_n \int x^n f(x)
\end{aligned} \tag{103}
$$

For example, consider the first few values of n. Using integration by parts multiple times, we obtain

$$
\begin{aligned}
&\int x f(x) = x \int f(x) - \iint f(x) \\
&\int x^2 f(x) = x^2 \int f(x) - 2x \iint f(x) + 2 \iiint f(x) \\
&\int x^3 f(x) = x^3 \int f(x) - 3x^2 \iint f(x) + 6x \iiint f(x) - 6 \iiiint f(x)
\end{aligned} \tag{104}
$$

In general, it can be shown using mathematical induction that

$$\int x^n f(x) = \sum_{k=0}^{n} (-1)^k \frac{\mathrm{d}^k x^n}{\mathrm{d}x^k} \int^{k+1} f(x) \tag{105}$$

Recall the formula for the repeated derivative of a power function:

$$\frac{\mathrm{d}^k x^n}{\mathrm{d}x^k} = (n)_k x^{n-k} \tag{106}$$

where $(n)_k$ is the Pochhammer symbol:

$$(n)_k = \frac{n!}{(n-k)!} \tag{107}$$

Substituting this into our expression yields

$$\int x^n f(x) = \sum_{k=0}^{n} (-1)^k (n)_k x^{n-k} \int^{k+1} f(x) \tag{108}$$

Now we check our solution. The Cauchy formula for repeated integration is

$$\int^{k+1} f(x) = \frac{1}{k!} \int_a^x \mathrm{d}t\, (x-t)^k f(t) \tag{109}$$

Hence we obtain

$$\begin{aligned} \int x^n f(x) &= \sum_{k=0}^{n} (-1)^k \frac{(n)_k}{k!} x^{n-k} \int_a^x \mathrm{d}t\, (x-t)^k f(t) \\ &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^{n-k} \int_a^x \mathrm{d}t\, (x-t)^k f(t) \end{aligned} \tag{110}$$

where $\binom{n}{k}$ is the binomial coefficient:

$$\binom{n}{k} = \frac{(n)_k}{k!} = \frac{n!}{k!(n-k)!} \tag{111}$$

Moving the integral outside yields

$$\begin{aligned} \int x^n f(x) &= \int_a^x \mathrm{d}t \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^{n-k} (x-t)^k f(t) \\ &= \int_a^x \mathrm{d}t\, f(t) \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^{n-k} (x-t)^k \\ &= \int_a^x \mathrm{d}t\, f(t) \sum_{k=0}^{n} \binom{n}{k} x^{n-k} (t-x)^k \end{aligned} \tag{112}$$

By the binomial theorem,

$$\sum_{k=0}^{n} \binom{n}{k} x^{n-k}(t-x)^k \\ = (x+t-x)^n \\ = t^n \tag{113}$$

Hence our expression further simplifies to

$$\int x^n f(x) = \int_a^x \mathrm{d}t\, f(t) t^n \tag{114}$$

which brings us back to the original expression on the left-hand side of the equation. Can this be generalized to non-integral values of n?

14 The fractional calculus

Is there such a thing as a square root of the derivative operator? This operator, which we will denote by D, would have to satisfy the operator equation

$$D^2 = \partial_x \tag{115}$$

That is, D would be an operator that, when applied twice to an expression, yields its derivative. We can generalize this further to an arbitrary nth root of ∂_x that yields the derivative when applied n times.

Earlier we encountered the formula for the repeated derivative of a power function:

$$\frac{\mathrm{d}^k x^n}{\mathrm{d}x^k} = (n)_k x^{n-k} = \frac{n!}{(n-k)!} x^{n-k} \tag{116}$$

We can readily generalize this to arbitrary values of k by using the gamma function, which generalizes the factorial function to non-integer values:

$$k! = \Gamma(k+1) \tag{117}$$

whenever k is an integer. Thus

$$\frac{\mathrm{d}^k x^n}{\mathrm{d}x^k} = \frac{\Gamma(k+1)}{\Gamma(n-k+1)} x^{n-k} \tag{118}$$

gives us the kth derivative of a power function, where k can be a non-integer value. This allows us to find the fractional derivatives of an arbitrary polynomial, which is a sum of power functions, by linearity.

Earlier we encountered the Cauchy formula for repeated integration, namely

$$\int^{k+1} f(x) = \frac{1}{k!} \int_a^x \mathrm{d}t\, (x-t)^k f(t) \tag{119}$$

This expression can be readily generalized to non-integer values of k using the gamma function. Thus we have

$$\int^{k+1} f(x) = \frac{1}{\Gamma(k+1)} \int_a^x \mathrm{d}t\,(x-t)^k f(t) \tag{120}$$

Equivalently,

$$\int^{k} f(x) = \frac{1}{\Gamma(k)} \int_a^x \mathrm{d}t\,(x-t)^{k-1} f(t) \tag{121}$$

Switching the sign of k yields

$$\begin{aligned} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k f(x) &= \frac{1}{\Gamma(-k)} \int_a^x \mathrm{d}t\,(x-t)^{-k-1} f(t) \\ &= \frac{1}{\Gamma(-k)} \int_a^x \frac{f(t)\mathrm{d}t}{(x-t)^{k+1}} \\ &= \frac{(-1)^{k+1}}{\Gamma(-k)} \int_a^x \frac{f(t)\mathrm{d}t}{(t-x)^{k+1}} \end{aligned} \tag{122}$$

which bears a close resemblance to the Cauchy integral formula

$$f^{(k)}(x) = \frac{k!}{2\pi i} \oint \frac{f(t)\mathrm{d}t}{(t-x)^{k+1}} \tag{123}$$

Why is this?

15 Generalized Stokes theorem

The fundamental theorem of calculus, divergence theorem, and Green's theorem can all be seen as special cases of a remarkably elegant and powerful theorem called the generalized Stokes theorem.

The theorem states that the integral of the exterior derivative of a differential form f over a manifold M is simply equal to the integral over the boundary of the manifold of the form itself:

$$\int_M \mathrm{d}f = \int_{\partial M} f \tag{124}$$

As one writer put it, the theorem reads almost like a Zen kōan: what happens on the outside is a function of the change within. The exterior derivative operator d and the boundary operator ∂ are, in a sense, adjoints.

To give a concrete example of the application of this theorem, we can derive the fundamental theorem of calculus:

$$
\begin{aligned}
\int_a^b f'(x)\mathrm{d}x &= \int_{[a,b]} \frac{\mathrm{d}f}{\mathrm{d}x}\mathrm{d}x \\
&= \int_{[a,b]} \mathrm{d}f \\
&= \int_{\partial[a,b]} f \\
&= f|_b - f|_a \\
&= f(b) - f(a)
\end{aligned} \tag{125}
$$

Similarly for the gradient theorem

$$
\begin{aligned}
\int_{\gamma[p,q]} \nabla f \cdot \mathrm{d}r &= \int_{\gamma[p,q]} \mathrm{d}f \\
&= \int_{\partial\gamma[p,q]} f \\
&= f(q) - f(p)
\end{aligned} \tag{126}
$$

16 The Newtonian potential

The Newtonian potential is an operator which, in a sense, acts like an "inverse" to the Laplacian operator. It is defined by a convolution with the Newtonian kernel Γ, which is the fundamental solution to the Laplace equation

$$\nabla^2 f = 0 \tag{127}$$

The Newtonian kernel is defined in dimension d as follows:

$$\Gamma(x) = \begin{cases} \frac{1}{2\pi}\log|x| & d = 2 \\ \frac{1}{d(2-d)\omega_d}|x|^{2-d} & d \neq 2 \end{cases} \tag{128}$$

where ω_d is the volume of the unit d-ball. The Newtonian potential is then

$$u(x) = \Gamma \star f(x) = \int \Gamma(x-y)f(y)\mathrm{d}y \tag{129}$$