Mechanics review

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1 Vectors and scalars

A vector is a geometric object that has magnitude and direction. Examples of vectors include position, displacement, velocity, acceleration, force, linear momentum, angular momentum.

A scalar is a quantity that only has magnitude but no direction. Examples include temperature, mass, charge, volume, time, speed, and energy.

1.1 Axioms of a vector space

Vectors can be added together and multiplied by scalars. They satisfy the following axioms:

$$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w} \quad \text{associativity of addition} \tag{1}$$

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u} \quad \text{commutativity of addition} \tag{2}$$

$$\exists \mathbf{0} \forall : \mathbf{v} + \mathbf{0} = \mathbf{v} \quad \text{identity element of addition} \tag{3}$$

$$\exists -\mathbf{v} : \mathbf{v} + -\mathbf{v} = \mathbf{0} \quad \text{inverse elements of addition} \tag{4}$$

$$a(b\mathbf{v}) = (ab)\mathbf{v} \quad \text{scalar and field multiplication} \tag{5}$$

$$1\mathbf{v} = \mathbf{v} \quad \text{identity element of scalar multiplication} \tag{6}$$

$$a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v} \quad \text{scalar multiplication and vector addition} \tag{7}$$

$$(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v} \quad \text{scalar multiplication and field addition} \tag{8}$$

1.2 The Levi Civita symbol

The Levi Civita symbol (also known as the permutation, antisymmetric, or alternating symbol) represents a collection of numbers, defined from the sign of a permutation of the natural numbers $1, 2, \ldots, n$ for some natural number n.

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is } (1,2,3),\ (2,3,1), \text{ or } (3,1,2) \\ -1 & \text{if } (i,j,k) \text{ is } (3,2,1),\ (2,1,3), \text{ or } (1,3,2) \\ 0 & \text{if } i = j \text{ or } j = k \text{ or } k = i \end{cases} \tag{9}$$

i.e. ε_{ijk} is 1 if (i, j, k) is an even permutation of $(1, 2, 3)$, -1 if it is an odd permutation, and 0 if any index is repeated.

The cross product can be defined in terms of the Levi-Civita symbol ε_{ijk} and an inner product η^{mi} (which is simply δ^{mi} for an orthonormal basis, where $\delta^{mi} = 0$ if $m \neq i$ and $\delta^{mi} = 1$ if $m = i$):

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} \Longleftrightarrow \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \eta^{mi} \varepsilon_{ijk} a^j b^k \tag{10}$$

Some useful identities in three dimensions are

$$\varepsilon_{ijk} \varepsilon^{imn} = \delta_j^m \delta_k^n - \delta_j^n \delta_k^m \tag{11}$$

$$\varepsilon_{jmn} \varepsilon^{imn} = 2\delta_j^i \tag{12}$$

$$\varepsilon_{ijk} \varepsilon^{ijk} = 6 \tag{13}$$

2 Coordinate systems

A coordinate system is a system which uses one or more numbers, or coordinates, to uniquely determine the position of a point in Euclidean space. A typical example of a coordinate system is the Cartesian coordinate system.

2.1 Cylindrical coordinates

Another common coordinate system is the cylindrical coordinate system, which specifies points by their distance ρ from a chosen reference axis, their direction ϕ from the axis relative to a chosen reference direction, and their (signed) distance z from a chosen reference plane perpendicular to the axis.

The conversion from polar coordinates (ρ, ϕ, z) to Cartesian coordinates (x, y, z) is

$$x = \rho \cos \phi \tag{14}$$

$$y = \rho \sin \phi \tag{15}$$

$$z = z \tag{16}$$

The conversion from Cartesian coordinates to cylindrical coordinates is

$$\rho = \sqrt{x^2 + y^2} \tag{17}$$

$$\phi = \arctan 2(y, x) \tag{18}$$

$$z = z \tag{19}$$

where $\arctan 2(y, x)$ is the angle in radians between the positive x-axis of a plane and the point given by the coordinates (x, y) on it. Cylindrical coordinates are useful in problems that have rotational symmetry.

2.2 Spherical coordinates

A spherical coordinate system is a coordinate system where the position of a point is specified by the radial distance r of that point from a fixed origin, its polar angle θ measured from a fixed zenith direction, and the azimuthal angle ϕ of its projection onto a reference plane that passes through the origin and is orthogonal to the zenith, measured from a fixed reference direction on that plane.

The conversion from spherical coordinates (r, θ, ϕ) to Cartesian coordinates (x, y, z) is given by

$$x = r \sin\theta \cos\phi \tag{20}$$
$$y = r \sin\theta \sin\phi \tag{21}$$
$$z = r \cos\theta \tag{22}$$

The conversion from Cartesian coordinates to spherical coordinates is

$$r = \sqrt{x^2 + y^2 + z^2} \tag{23}$$
$$\theta = \arccos\left(\frac{z}{r}\right) \tag{24}$$
$$\phi = \arctan 2(y, x) \tag{25}$$

Spherical coordinates are useful in problems that have spherical symmetry.

2.3 Derivatives in polar coordinates

Using $x = r\cos\phi$ and $y = r\sin\phi$

$$\begin{aligned} r\frac{\partial f}{\partial r} &= r\frac{\partial x}{\partial r}\frac{\partial f}{\partial x} + r\frac{\partial y}{\partial r}\frac{\partial f}{\partial y} \\ &= r\cos\phi\frac{\partial f}{\partial x} + r\sin\phi\frac{\partial f}{\partial y} \\ &= x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} \end{aligned} \tag{26}$$

$$\begin{aligned} \frac{\partial f}{\partial \phi} &= \frac{\partial x}{\partial \phi}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial \phi}\frac{\partial f}{\partial y} \\ &= -r\sin\phi\frac{\partial f}{\partial x} + r\cos\phi\frac{\partial f}{\partial y} \\ &= -y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y} \end{aligned} \tag{27}$$

These two results are equivalent to

$$r\frac{\partial}{\partial r} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \tag{28}$$

$$\frac{\partial}{\partial \phi} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \tag{29}$$

The basis vectors of a polar coordinate system are $\mathbf{e_r} = (\cos\phi, \sin\phi)$ and $\mathbf{e}_\phi = (-\sin\phi, \cos\phi)$. Therefore

$$\begin{aligned}
\mathrm{d}\mathbf{e_r} &= \mathrm{d}(\cos\phi, \sin\phi) \\
&= (\mathrm{d}\cos\phi, \mathrm{d}\sin\phi) \\
&= (-\sin\phi \mathrm{d}\phi, \cos\phi \mathrm{d}\phi) \\
&= (-\sin\phi, \cos\phi)\mathrm{d}\phi \\
&= \mathbf{e}_\phi \mathrm{d}\phi
\end{aligned} \tag{30}$$

In dot notation, this is equivalent to

$$\dot{\mathbf{e_r}} = \mathbf{e}_\phi \dot{\phi} \tag{31}$$

$$\begin{aligned}
\mathrm{d}\mathbf{e}_\phi &= \mathrm{d}(-\sin\phi, \cos\phi) \\
&= (-\mathrm{d}\sin\phi, \mathrm{d}\cos\phi) \\
&= (-\cos\phi \mathrm{d}\phi, -\sin\phi \mathrm{d}\phi) \\
&= (-\cos\phi, -\sin\phi)\mathrm{d}\phi \\
&= -(\cos\phi, \sin\phi)\mathrm{d}\phi \\
&= -\mathbf{e_r}\mathrm{d}\phi
\end{aligned} \tag{32}$$

In dot notation, this is equivalent to

$$\dot{\mathbf{e}_\phi} = -\mathbf{e_r}\dot{\phi} \tag{33}$$

Using the previous identities, we find that

$$\mathbf{r} = r\mathbf{e_r} \tag{34}$$

$$\begin{aligned}
\mathrm{d}\mathbf{r} &= \mathrm{d}(r\mathbf{e_r}) \\
&= (\mathrm{d}r)\mathbf{e_r} + r(\mathrm{d}\mathbf{e_r}) \\
&= (\mathrm{d}r)\mathbf{e_r} + r(\mathrm{d}\phi)\mathbf{e}_\phi
\end{aligned} \tag{35}$$

In dot notation, this is equivalent to

$$\dot{\mathbf{r}} = \dot{r}\mathbf{e_r} + r\dot{\phi}\mathbf{e}_\phi \tag{36}$$

$$\begin{aligned}
\mathrm{d}^2\mathbf{r} &= \mathrm{d}(\mathrm{d}\mathbf{r}) \\
&= \mathrm{d}((\mathrm{d}r)\mathbf{e_r} + r(\mathrm{d}\phi)\mathbf{e}_\phi) \\
&= \mathrm{d}((\mathrm{d}r)\mathbf{e_r}) + \mathrm{d}(r(\mathrm{d}\phi)\mathbf{e}_\phi) \\
&= (\mathrm{d}^2r)\mathbf{e_r} + (\mathrm{d}r)(\mathrm{d}\mathbf{e_r}) + (\mathrm{d}r)(\mathrm{d}\phi)\mathbf{e}_\phi + r(\mathrm{d}^2\phi)\mathbf{e}_\phi + r(\mathrm{d}\phi)(\mathrm{d}\mathbf{e}_\phi) \\
&= (\mathrm{d}^2r)\mathbf{e_r} + (\mathrm{d}r)\mathbf{e}_\phi(\mathrm{d}\phi) + (\mathrm{d}r)(\mathrm{d}\phi)\mathbf{e}_\phi + r(\mathrm{d}^2\phi)\mathbf{e}_\phi - r^2(\mathrm{d}\phi)\mathbf{e_r} \\
&= ((\mathrm{d}^2r) - r^2(\mathrm{d}\phi))\mathbf{e_r} + (r(\mathrm{d}^2\phi) + 2(\mathrm{d}r)(\mathrm{d}\phi))\mathbf{e}_\phi
\end{aligned} \tag{37}$$

In dot notation, this is equivalent to

$$\ddot{\mathbf{r}} = (\ddot{r} - r\dot{\phi}^2)\mathbf{e_r} + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\mathbf{e}_\phi \tag{38}$$

3 Non-inertial reference frames

Suppose we have a particle at position $\mathbf{x_A}$ in an inertial frame A. Consider a non-inertial frame B whose origin with respect to the inertial one is $\mathbf{x_{BA}}$. Let the position of the particle in frame B be $\mathbf{x_B}$.

Let the coordinate axes in B be represented by the unit vectors $\mathbf{u_j}$. Then

$$\mathbf{x_B} = x^j\mathbf{u_j} \tag{39}$$

where the Einstein summation convention is used. Then, the position of the particle in frame A is

$$\begin{aligned}
\mathbf{x_A} &= \mathbf{x_{BA}} + \mathbf{x_B} \\
&= \mathbf{x_{BA}} + x^j\mathbf{u_j}
\end{aligned} \tag{40}$$

Taking the time derivative yields

$$\begin{aligned}
\mathbf{v_A} &= \frac{\mathrm{d}\mathbf{x_A}}{\mathrm{d}t} \\
&= \frac{\mathrm{d}\mathbf{x_{BA}}}{\mathrm{d}t} + \frac{\mathrm{d}\mathrm{x}^\mathrm{j}\mathbf{u_j}}{\mathrm{d}t} \\
&= \frac{\mathrm{d}\mathbf{x_{BA}}}{\mathrm{d}t} + \frac{\mathrm{d}x^j}{\mathrm{d}t}\mathbf{u_j} + x^j\frac{\mathrm{d}\mathbf{u_j}}{\mathrm{d}t} \\
&= \mathbf{v_{BA}} + \mathbf{v_B} + x^j\frac{\mathrm{d}\mathbf{u_j}}{\mathrm{d}t}
\end{aligned} \tag{41}$$

Hence the velocity of the particle in frame A consists of what observers in frame B see the velocity to be ($\mathbf{v_B}$) and two extra terms related to the rate of change of frame B's coordinate axes (the first one being the velocity of the moving origin and the second one due to the fact that different locations in the non-inertial frame have different apparent velocities due to rotation of the frame.

To find the acceleration, we take another derivative:

$$\begin{aligned} \mathbf{a_A} &= \frac{\mathrm{d}\mathbf{v_A}}{\mathrm{d}t} \\ &= \frac{\mathrm{d}\mathbf{v_{BA}}}{\mathrm{d}t} + \frac{\mathrm{d}\mathbf{v_B}}{\mathrm{d}t} + \frac{\mathrm{d}x^j}{\mathrm{d}t}\frac{\mathrm{d}\mathbf{u_j}}{\mathrm{d}t} + x^j\frac{\mathrm{d}^2\mathbf{u_j}}{\mathrm{d}t^2} \\ &= \mathbf{a_{BA}} + \frac{\mathrm{d}\mathbf{v_B}}{\mathrm{d}t} + v^j\frac{\mathrm{d}\mathbf{u_j}}{\mathrm{d}t} + x^j\frac{\mathrm{d}^2\mathbf{u_j}}{\mathrm{d}t^2} \end{aligned} \tag{42}$$

The velocity derivative is

$$\begin{aligned} \frac{\mathrm{d}\mathbf{v_B}}{\mathrm{d}t} &= \frac{\mathrm{d}v^j\mathbf{u_j}}{\mathrm{d}t} \\ &= \frac{\mathrm{d}v^j}{\mathrm{d}t}\mathbf{u_j} + v^j\frac{\mathrm{d}\mathbf{u_j}}{\mathrm{d}t} \\ &= a^j\mathbf{u_j} + v^j\frac{\mathrm{d}\mathbf{u_j}}{\mathrm{d}t} \\ &= \mathbf{a_B} + v^j\frac{\mathrm{d}\mathbf{u_j}}{\mathrm{d}t} \end{aligned} \tag{43}$$

Hence

$$\mathbf{a_A} = \mathbf{a_{BA}} + \mathbf{a_B} + 2v^j\frac{\mathrm{d}\mathbf{u_j}}{\mathrm{d}t} + x^j\frac{\mathrm{d}^2\mathbf{u_j}}{\mathrm{d}t^2} \tag{44}$$

Hence the acceleration of the particle in frame A consists of acceleration in frame B call the acceleration $\mathbf{a_B}$. In addition there are three acceleration terms related to the movement of frame B's coordinate axes: one term related to the acceleration of the origin of frame B, namely $\mathbf{a_{AB}}$, and two terms related to rotation of frame B.

Consequently, observers in B will see the particle motion as possessing extra acceleration and attribute it to "forces" acting on the particle, but which observers in A say are "fictitious" forces arising simply because the frame B is non-inertial.

In terms of forces, the accelerations are multiplied by the particle mass:

$$\mathbf{F_A} = \mathbf{F_B} + m\mathbf{a_{BA}} + 2mv^j\frac{\mathrm{d}\mathbf{u_j}}{\mathrm{d}t} + mx^j\frac{\mathrm{d}^2\mathbf{u_j}}{\mathrm{d}t^2} \tag{45}$$

Hence the force observed in frame B is

$$\begin{aligned} \mathbf{F_B} &= \mathbf{F_A} - m\mathbf{a_{BA}} - 2mv^j\frac{\mathrm{d}\mathbf{u_j}}{\mathrm{d}t} - mx^j\frac{\mathrm{d}^2\mathbf{u_j}}{\mathrm{d}t^2} \\ &= \mathbf{F_A} + \mathbf{F}_{\text{fictitious}} \end{aligned} \tag{46}$$

3.1 Rotating frame

Let $\mathbf{\Omega}$ be the angular velocity vector of frame B, with magnitude

$$|\mathbf{\Omega}| = \frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega \tag{47}$$

where $\omega(t)$ is the angular speed. Then the time derivative of a vector $\mathbf{u}$ in frame B is

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \mathbf{\Omega} \times \mathbf{u} \tag{48}$$

and

$$\begin{aligned} \frac{\mathrm{d}^2\mathbf{u}}{\mathrm{d}t^2} &= \frac{\mathrm{d}\mathbf{\Omega}}{\mathrm{d}t} \times \mathbf{u} + \mathbf{\Omega} \times \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} \\ &= \frac{\mathrm{d}\mathbf{\Omega}}{\mathrm{d}t} \times \mathbf{u} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{u}) \end{aligned} \tag{49}$$

Recall that

$$\mathbf{a_A} = \mathbf{a_{BA}} + \mathbf{a_B} + 2v^j \frac{\mathrm{d}\mathbf{u_j}}{\mathrm{d}t} + x^j \frac{\mathrm{d}^2\mathbf{u_j}}{\mathrm{d}t^2} \tag{50}$$

Letting $\mathbf{a_{BA}} = 0$ to remove translational acceleration,

$$\begin{aligned} \mathbf{a_A} &= \mathbf{a_B} + 2v^j \frac{\mathrm{d}\mathbf{u_j}}{\mathrm{d}t} + x^j \frac{\mathrm{d}^2\mathbf{u_j}}{\mathrm{d}t^2} \\ &= \mathbf{a_B} + 2v^j \mathbf{\Omega} \times \mathbf{u_j} + x^j \left(\frac{\mathrm{d}\mathbf{\Omega}}{\mathrm{d}t} \times \mathbf{u} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{u}) \right) \\ &= \mathbf{a_B} + 2v^j \mathbf{\Omega} \times \mathbf{u_j} + x^j \frac{\mathrm{d}\mathbf{\Omega}}{\mathrm{d}t} \times \mathbf{u} + x^j \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{u}) \\ &= \mathbf{a_B} + 2v^j \mathbf{\Omega} \times \mathbf{u_j} + \frac{\mathrm{d}\mathbf{\Omega}}{\mathrm{d}t} \times \mathbf{x_B} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{x_B}) \end{aligned} \tag{51}$$

The physical acceleration $\mathbf{a_A}$ due to real external forces on the object observed from frame A is not just the acceleration as seen from frame B but has several additional terms associated with the rotation B. As seen in the rotational frame, the acceleration is given by

$$\mathbf{a_B} = \mathbf{a_A} - 2\mathbf{\Omega} \times \mathbf{v_B} - \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{x_B}) - \frac{\mathrm{d}\mathbf{\Omega}}{\mathrm{d}t} \times \mathbf{x_B} \tag{52}$$

In terms of forces

$$m\mathbf{a_B} = m\mathbf{a_A} - 2m\mathbf{\Omega} \times \mathbf{v_B} - m\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{x_B}) - m\frac{\mathrm{d}\mathbf{\Omega}}{\mathrm{d}t} \times \mathbf{x_B} \tag{53}$$

Therefore

$$\mathbf{F_B} = \mathbf{F_A} + \mathbf{F}_{\text{fictitious}} \tag{54}$$

where

$$\mathbf{F}_{\text{fictitious}} = -2m\mathbf{\Omega} \times \mathbf{v_B} - m\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{x_B}) - m\frac{\mathrm{d}\mathbf{\Omega}}{\mathrm{d}t} \times \mathbf{x_B} \tag{55}$$

3.2 Local coordinate system on Earth

The vector formula for the Coriolis acceleration is

$$\mathbf{a_C} = -2\mathbf{\Omega} \times \mathbf{v} \tag{56}$$

Multiplying both sides by mass yields the Coriolis force:

$$\mathbf{F_C} = -2m\mathbf{\Omega} \times \mathbf{v} \tag{57}$$

Consider a location with latitude λ. A local coordinate system is set up with the x axis pointing east, the y axis pointing north, and the z axis pointing upward. The angular velocity vector of the Earth can be expressed in this local coordinate system as

$$\mathbf{\Omega} = \omega \begin{pmatrix} 0 \\ \cos\varphi \\ \sin\varphi \end{pmatrix} \tag{58}$$

The velocity is expressed in this local coordinate system as

$$\mathbf{v} = \begin{pmatrix} v_e \\ v_n \\ v_u \end{pmatrix} \tag{59}$$

Hence the Coriolis acceleration is

$$\mathbf{a_C} = -2\Omega \times \mathbf{v} = 2\omega \begin{pmatrix} v_n \sin\phi - v_u \cos\phi \\ -v_e \sin\phi \\ v_e \cos\phi \end{pmatrix} \tag{60}$$

3.3 Surface of water in rotating cylinder

Consider a rotating cylinder filled with liquid. It can be shown that the liquid surface will form a parabola (more accurately, a paraboloid).

There are two forces acting on a mass m at the surface of the liquid at coordinates x and y, where x is the horizontal distance from the axis of rotation and y is the height from some reference elevation.

These two forces are the centripetal force $F_{\text{centripetal}} = m\omega^2 x$, which acts horizontally, and the gravitational force $F_{\text{gravitational}} = mg$, which acts vertically. Therefore the components of the normal force acting on the mass are

$$F_y = F\cos\theta = mg \tag{61}$$

$$F_x = F\sin\theta = m\omega^2 x \tag{62}$$

where θ is the angle that the tangent to the curve makes with the horizontal axis (which is perpendicular to the angle that the *normal* to the curve makes with the horizontal axis). Therefore

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{F\sin\theta}{F\cos\theta} = \frac{m\omega^2 x}{mg} = \frac{\omega^2 x}{g} \tag{63}$$

Integrating this with respect to x yields

$$y = \int \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = \int \frac{\omega^2 x}{g}\mathrm{d}x = \frac{\omega^2 x^2}{2g} + C \tag{64}$$

where C is some constant.

4 Oscillators

4.1 Simple harmonic oscillators

A simple harmonic oscillator is an oscillator that is neither damped nor driven. It consists of a mass m which experiences a single force F which pulls the mass in the direction of the equilibrium point and depends only on the mass's displacement from equilibrium x and a constant k. The equations of motion for the system are

$$F = ma = m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = m\ddot{x} = -kx \tag{65}$$

Solving this differential equation yields

$$x = A\cos(\omega t + \phi) \tag{66}$$

where

$$\omega = \sqrt{\frac{k}{m}} = \frac{2\pi}{T} \tag{67}$$

ω is the angular frequency of the oscillator and T is the period of the oscillator. The potential energy stored in a simple harmonic oscillator at displacement x is

$$U = \frac{1}{2}kx^2 \tag{68}$$

4.2 Springs in parallel

For springs in parallel, whatever distance spring 1 is compressed has to be the same amount spring 2 is compressed.

The force on the block attached to the springs is then

$$F = F_1 + F_2 = -k_1 x - k_2 x = -(k_1 + k_2)x \tag{69}$$

where k_1 and k_2 are the spring constants of spring 1 and spring 2, respectively. Hence the equivalent spring constant is

$$k_{\text{equivalent}} = k_1 + k_2 \tag{70}$$

This is the same as the formula for the effective conductance (reciprocal of resistance) of two resistors G_1 and G_2 in parallel:

$$G_{\text{equivalent}} = G_1 + G_2 \tag{71}$$

4.3 Springs in series

Consider two springs placed in series, with a mass attached to the end of the second. ach of the springs will experience corresponding displacements x1 and x2 for a total displacement of x1 + x2. We are looking for an effective or equivalent spring constant such that

$$F = -k_{\text{equivalent}}(x_1 + x_2) \tag{72}$$

The force that each spring experiences is the same, otherwise the springs would buckle. Therefore

$$F = -k_1 x_1 = -k_2 x_2 \tag{73}$$

Therefore, solving for x_1 in terms of x_2,

$$x_1 = \frac{k_2}{k_1} x_2 \tag{74}$$

Substituting this into our previous equation yields

$$F = -k_{\text{equivalent}} \left(\frac{k_2}{k_1} x_2 + x_2 \right) = -k_{\text{equivalent}} \left(\frac{k_2 + k_1}{k_1} \right) x_2 \tag{75}$$

The force that each spring experiences is also the same as that experienced by the block, so

$$F_2 = -k_2 x_2 = -k_{\text{equivalent}} \left(\frac{k_2 + k_1}{k_1} \right) x_2 = F \tag{76}$$

Canceling $-x_2$ on both sides yields

$$k_2 = k_{\text{equivalent}} \left(\frac{k_2 + k_1}{k_1} \right) \tag{77}$$

or

$$\frac{k_2 k_1}{k_2 + k_1} = k_{\text{equivalent}} \tag{78}$$

which can also be written as

$$\frac{1}{k_2} + \frac{1}{k_1} = \frac{1}{k_{\text{equivalent}}} \tag{79}$$

This is the same as the formula for the effective conductance of two resistors G_1 and G_2 in series:

$$\frac{1}{G_{\text{equivalent}}} = \frac{1}{G_1} + \frac{1}{G_2} \tag{80}$$

4.4 Damped harmonic oscillator

In a damped harmonic oscillator, friction, or damping, slows the motion of the system. The velocity decreases due to a frictional force which can be modeled as proportional to the velocity v of the object by a factor of a damping coefficient c:

$$F_{\text{frictional}} = -cv \tag{81}$$

The equations of motion are then

$$F = m\ddot{x} = F_{\text{spring}} + F_{\text{damping}} = -kx - c\dot{x} \tag{82}$$

If there are no external forces, $F_{\text{external}} = 0$ and the previous equation can be rewritten in the form

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0 \tag{83}$$

or

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = 0 \tag{84}$$

where $\omega_0 = \sqrt{\frac{k}{m}}$ is the undamped angular frequency of the oscillator and $\zeta = \frac{c}{2\sqrt{mk}}$ is the damping ratio. The damping ratio determines the overall behavior of the system.

If $\zeta < 1$, the system oscillates while the amplitude gradually decreases to zero (as long as $\zeta > 0$). The angular frequency of the underdamped harmonic oscillator is given by

$$\omega_1 = \omega_0\sqrt{1 - \zeta^2} \tag{85}$$

The exponential decay of the amplitude of an underdamped harmonic oscillator is given by

$$\lambda = \omega_0 \zeta \tag{86}$$

meaning the decay time $\tau = 1/(\zeta\omega_0)$.

If $\zeta = 1$, the system is critically damped and returns to steady state as quickly as possible without oscillating.

If $\zeta > 1$, the system is overdamped and returns to steady state without oscillating with exponentially decaying amplitude.

The total energy of a damped oscillator is the sum of its kinetic and potential energies:

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega_0^2 x^2 \tag{87}$$

Differentiating the above equation yields

$$\dot{E} = m\dot{x}\ddot{x} + m\omega_0^2 x\dot{x} = m\dot{x}(\ddot{x} + \omega_0^2 x) \tag{88}$$

But since

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0 \tag{89}$$

Then

$$\dot{E} = -2m\beta\dot{x}^2 \tag{90}$$

4.5 The quality factor

The energy loss rate of a weakly damped oscillator ($\beta \ll \omega_0$ or $\zeta \ll 1$) can be characterized in terms of a parameter Q called the quality factor. The quality factor is defined by

$$Q = 2\pi \times \frac{\text{Energy stored}}{\text{Energy lost per cycle}} \tag{91}$$

If the oscillator is weakly damped then the energy lost per period is relatively small and Q is therefore much larger than unity. Roughly speaking Q is the number of oscillations that the oscillator typically completes after being set in motion before its amplitude decays to a negligible value.

The most general solution for a weakly damped oscillator can be written as

$$x = Ae^{-\beta t}\cos(\omega_1 t - \phi) \tag{92}$$

Hence

$$\dot{x} = -A(\beta e^{-\beta t}\cos(\omega_1 t - \phi) + \omega_1 e^{-\beta t}\sin(\omega_1 t - \phi)) \tag{93}$$

Therefore the energy lost during a single oscillation period is

$$\Delta E = -\int_0^T \dot{E}\mathrm{d}t = \int_0^T 2m\beta\dot{x}^2\mathrm{d}t = 2m\beta\int_0^T \dot{x}^2\mathrm{d}t \tag{94}$$

Substituting the previous expression for $\dot{x}$ yields

$$\begin{aligned}\Delta E &= 2m\beta\int_0^T A^2e^{-2\beta t}(\beta e^{-\beta t}\cos(\omega_1 t-\phi)+\omega_1 e^{-\beta t}\sin(\omega_1 t-\phi))^2\mathrm{d}t \\ &= 2m\beta A^2\int_0^{\frac{2\pi}{\omega_1}} e^{-2\beta t}(\beta e^{-\beta t}\cos(\omega_1 t-\phi)+\omega_1 e^{-\beta t}\sin(\omega_1 t-\phi))^2\mathrm{d}t\end{aligned} \tag{95}$$

In the weakly damped limit, $\beta \ll \omega_1$ so the exponential factor is approximately unity in the interval, so that

$$\begin{aligned}\Delta E &\approx 2m\beta A^2\int_0^{\frac{2\pi}{\omega_1}} (\beta e^{-\beta t}\cos(\omega_1 t-\phi)+\omega_1 e^{-\beta t}\sin(\omega_1 t-\phi))^2\mathrm{d}t \\ &= \frac{2m\beta x_0^2}{\omega_1}\int_0^{2\pi}(\beta^2\cos^2\theta + 2\beta\omega_1\cos\theta\sin\theta + \omega_1^2\sin^2\theta)\mathrm{d}\theta \\ &= \frac{2m\beta x_0^2}{\omega_1}\left(\beta^2\int_0^{2\pi}\cos^2\theta\mathrm{d}\theta + 2\beta\omega_1\int_0^{2\pi}\cos\theta\sin\theta\mathrm{d}\theta + \omega_1^2\sin^2\theta\mathrm{d}\theta\right)\end{aligned} \tag{96}$$

Both $\cos^2\theta$ and $\sin^2\theta$ have average values of $\frac{1}{2}$ in the interval between 0 and 2π, whereas $\cos\theta\sin\theta$ has an average value of zero. Hence

$$\Delta E \approx \frac{2m\beta A^2}{\omega_1}(\beta^2\pi+\omega_1^2\pi) = \frac{2\pi m\beta A^2}{\omega_1}(\beta^2+\omega_1^2) = 2\pi m\omega_0^2A^2\frac{\beta}{\omega_1} \tag{97}$$

The energy stored in the oscillator at $t = 0$ is

$$E = \frac{1}{2}m\omega_0^2A^2 \tag{98}$$

Therefore

$$Q = 2\pi\frac{E}{\Delta E} = 2\pi\frac{\frac{1}{2}m\omega_0^2A^2}{2\pi m\omega_0^2A^2\frac{\beta}{\omega_1}} = \frac{\omega_1}{2\beta} = \frac{\sqrt{\omega_0^2-\beta^2}}{2\beta} \tag{99}$$

But since $\beta \ll \omega_0$, this reduces to

$$Q \approx \frac{\omega_0}{2\beta} = \frac{1}{2\zeta} \tag{100}$$

5 Phase space

Because the equation of motion is of second order, the state of motion of a one-dimensional oscillator is completely specified as a function of time if two quantities are given at one instant of time, that is, the initial conditions $x(t_0)$ and $\dot{x}(t_0)$. We may consider the quantities x and $\dot{x}$ to be the coordinates of a point in a two-dimensional space, called phase space. As time evolves, the point $(x, \dot{x})$ describing the state of the oscillating particle will move along a certain phase path in phase space. The combination of all possible phase paths constitutes the phase portrait or the phase diagram of the oscillator. We know that

$$x = A\sin(\omega_0 t - \phi) = A\sin\theta \tag{101}$$

$$\dot{x} = A\omega_0\cos(\omega_0 t - \phi) = A\omega_0\cos\theta \tag{102}$$

Then

$$x^2 + \frac{\dot{x}^2}{\omega_0^2} = A^2 \tag{103}$$

Because the total energy E of the oscillator is $\frac{1}{2}kA^2$ and because $\omega_0^2 = \frac{k}{m}$, this can be rewritten as

$$\frac{kx^2}{2E} + \frac{m\dot{x}^2}{2E} = 1 \tag{104}$$

Each phase path then corresponds to a definite total energy of the oscillator. Since

$$\ddot{x} + \omega_0^2 x = 0 \tag{105}$$

We can replace this with a pair of equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \dot{x} \tag{106}$$

$$\frac{\mathrm{d}\dot{x}}{\mathrm{d}t} = -\omega_0^2 x \tag{107}$$

Dividing the second equation by the first yields

$$\frac{\mathrm{d}\dot{x}}{\mathrm{d}x} = -\omega_0^2 \frac{x}{\dot{x}} \tag{108}$$

The solution of which is an ellipse in phase space.

5.1 Addendum: Complete derivation for rotating frames

Primed variables correspond to the inertial, non-rotating reference frame, while unprimed variables correspond to the non-inertial, rotating reference frame.

$$
\begin{aligned}
r' &= R + r \\
\mathrm{d}r' &= \mathrm{d}R + \mathrm{d}r \\
&= \mathrm{d}R + \mathrm{d}(r^i \mathbf{e_i}) \\
&= \mathrm{d}R + (\mathrm{d}r^i)\mathbf{e_i} + r^i(\mathrm{d}\mathbf{e_i}) \\
&= \mathrm{d}R + v^i \mathbf{e_i} + r^i \mathrm{d}\mathbf{e_i} \\
&= \mathrm{d}R + v + r^i \mathrm{d}\mathbf{e_i} \\
\mathrm{d}\mathbf{e_i} &= \omega \times \mathbf{e_i} \\
\mathrm{d}r' &= \mathrm{d}R + v + r^i(\omega \times \mathbf{e_i}) \\
&= \mathrm{d}R + v + \omega \times r^i \mathbf{e_i} \\
&= \mathrm{d}R + v + \omega \times r \\
\mathrm{d}^2 r' &= \mathrm{d}^2 R + \mathrm{d}v + \mathrm{d}(\omega \times r) \\
&= \mathrm{d}^2 R + \mathrm{d}v + (\mathrm{d}\omega) \times r + \omega \times (\mathrm{d}r) \\
&= \mathrm{d}^2 R + \mathrm{d}v + (\mathrm{d}\omega) \times r + \omega \times (v + \omega \times r) \\
&= \mathrm{d}^2 R + \mathrm{d}v + (\mathrm{d}\omega) \times r + \omega \times v + \omega \times (\omega \times r) \\
&= \mathrm{d}^2 R + \mathrm{d}(v^i \mathbf{e_i}) + (\mathrm{d}\omega) \times r + \omega \times v + \omega \times (\omega \times r) \\
&= \mathrm{d}^2 R + (\mathrm{d}v^i)\mathbf{e_i} + v^i(\mathrm{d}\mathbf{e_i}) + (\mathrm{d}\omega) \times r + \omega \times v + \omega \times (\omega \times r) \\
&= \mathrm{d}^2 R + a^i \mathbf{e_i} + v^i(\mathrm{d}\mathbf{e_i}) + (\mathrm{d}\omega) \times r + \omega \times v + \omega \times (\omega \times r) \\
&= \mathrm{d}^2 R + a + v^i(\mathrm{d}\mathbf{e_i}) + (\mathrm{d}\omega) \times r + \omega \times v + \omega \times (\omega \times r) \\
&= \mathrm{d}^2 R + a + v^i(\omega \times \mathbf{e_i}) + (\mathrm{d}\omega) \times r + \omega \times v + \omega \times (\omega \times r) \\
&= \mathrm{d}^2 R + a + \omega \times v^i \mathbf{e_i} + (\mathrm{d}\omega) \times r + \omega \times v + \omega \times (\omega \times r) \\
&= \mathrm{d}^2 R + a + \omega \times v + (\mathrm{d}\omega) \times r + \omega \times v + \omega \times (\omega \times r) \\
&= \mathrm{d}^2 R + a + 2\omega \times v + (\mathrm{d}\omega) \times r + \omega \times (\omega \times r) \\
m\ddot{r}' &= m\ddot{R} + ma + 2m\omega \times v + m\dot{\omega} \times r + m\omega \times (\omega \times r) \\
F' &= m\ddot{R} + F + 2m\omega \times v + m\dot{\omega} \times r + m\omega \times (\omega \times r) \\
F &= F' - m\ddot{R} - 2m\omega \times v - m\dot{\omega} \times r - m\omega \times (\omega \times r)
\end{aligned}
$$

The first fictitious force is the Coriolis force, the second fictitious force is the Euler force, and the third fictitious force is the centrifugal force.