1 Preliminary

1.1 Dirichlet condition

1.1.1 Wave equation

Consider the wave equation

$$u_{tt} = c^2 u_{xx} \tag{1}$$

with homogeneous Dirichlet boundary conditions

$$u(0,t) = 0 \tag{2}$$
$$u(l,t) = 0 \tag{3}$$

and initial conditions

$$u(x,0) = \phi(x) \tag{4}$$
$$u_t(x,0) = \psi(x) \tag{5}$$

Let

$$u(x,t) = X(x)T(t) \tag{6}$$

Substituting this into the wave equation yields

$$(X(x)T(t))_{tt} = c^2(X(x)T(t))_{xx} \tag{7}$$
$$X(x)T''(t) = c^2 X''(x)T(t) \tag{8}$$

Hence

$$\frac{T''}{c^2 T} = \frac{X''}{X} = -\lambda \tag{9}$$

where λ is a constant, since $\lambda_x = \lambda_t = 0$. Hence we obtain a pair of separate ordinary differential equations

$$X'' + \lambda X = 0 \tag{10}$$
$$T'' + c^2 \lambda T = 0 \tag{11}$$

The solutions are

$$T(t) = A\cos tc\sqrt{\lambda} + B\sin tc\sqrt{\lambda} \tag{12}$$
$$X(x) = C\cos x\sqrt{\lambda} + D\sin x\sqrt{\lambda} \tag{13}$$

The boundary conditions require that $X(0) = X(l) = 0$. Thus

$$X(0) = C = 0 \tag{14}$$
$$X(l) = D\sin l\sqrt{\lambda} = 0 \tag{15}$$

For the sine term to be zero, $l\sqrt{\lambda_n} = n\pi$ for $n \in \mathbb{Z}$. Hence

$$\sqrt{\lambda_n} = \frac{n\pi}{l} \tag{16}$$
$$\lambda_n = \left(\frac{n\pi}{l}\right)^2 \tag{17}$$
$$X_n(x) = \sin x\lambda_n = \sin\frac{n\pi x}{l} \tag{18}$$

Therefore the separated solutions for each n are

$$\begin{aligned} u_n(x,t) &= (A_n\cos ct\sqrt{\lambda} + B_n\sin ct\sqrt{\lambda})\sin x\sqrt{\lambda} \\ &= \left(A_n\cos\frac{n\pi ct}{l} + B_n\sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l} \end{aligned} \tag{19}$$

Therefore

$$\begin{aligned} u(x,t) &= \sum_n u_n(x,t) \\ &= \sum_n \left(A_n\cos\frac{n\pi ct}{l} + B_n\sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l} \end{aligned} \tag{20}$$

From the boundary conditions we have

$$\phi(x) = u(x,0) = \sum_n A_n \sin\frac{n\pi x}{l} \tag{21}$$
$$\psi(x) = u_t(x,0) = \sum_n \frac{n\pi c}{l} B_n \sin\frac{n\pi x}{l} \tag{22}$$

1.1.2 Diffusion

$$u_t = ku_{xx} \tag{23}$$
$$u(0,t) = 0 \tag{24}$$
$$u(l,t) = 0 \tag{25}$$
$$u(x,0) = \phi(x) \tag{26}$$

Letting $u = T(t)X(x)$:

$$\frac{T'}{kT} = \frac{X''}{X} = -\lambda \tag{27}$$

Therefore

$$T' + \lambda kT = 0 \tag{28}$$
$$X'' + \lambda X = 0 \tag{29}$$

Therefore

$$T(t) = A\exp(-\lambda kt) \tag{30}$$
$$X(x) = C\cos x\sqrt{\lambda} + D\sin x\sqrt{\lambda} \tag{31}$$

Since $X(0) = X(l) = 0$

$$X(0) = C = 0 \tag{32}$$
$$X(l) = D\sin l\sqrt{\lambda} = 0 \tag{33}$$

Therefore, as before,

$$u(x,t) = \sum_n A_n \mathrm{e}^{-(n\pi/l)^2 kt} \sin \frac{n\pi x}{l} \tag{34}$$

From the boundary condition,

$$\phi(x) = u(x,0) = \sum_n A_n \sin \frac{n\pi x}{l} \tag{35}$$

Hence our solution is expressible as a sine series if the initial data are.

1.2 Neumann condition

The Neumann condition is

$$u_x(0,t) = 0 \tag{36}$$
$$u_x(l,t) = 0 \tag{37}$$

Then the eigenfunctions are the solutions $X(x)$ of

$$X'' + \lambda X = 0 \tag{38}$$
$$X'(0) = 0 \tag{39}$$
$$X'(l) = 0 \tag{40}$$

2 Separation of variables

2.1 Dirichlet condition

2.1.1 Diffusion equation

Let

$$u_t = ku_{xx} \tag{41}$$
$$u(0,t) = u(l,t) = 0 \tag{42}$$
$$u(x,0) = \phi(x) \tag{43}$$

then

$$u(x,t) = \sum_n A_n \mathrm{e}^{-(n\pi/l)^2 kt} \sin \frac{n\pi x}{l} \tag{44}$$

where

$$\phi(x) = \sum_n A_n \sin \frac{n\pi x}{l} \tag{45}$$

2.1.2 Wave eqaution

Let

$$u_{tt} = c^2 u_{xx} \tag{46}$$
$$u(0,t) = u(l,t) = 0 \tag{47}$$
$$u(x,0) = \phi(x) \tag{48}$$
$$u_t(x,0) = \psi(x) \tag{49}$$

then

$$u(x,t) = \sum_n \left(A_n \cos \frac{n\pi ct}{l} + B_n \sin \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l} \tag{50}$$

where

$$\phi(x) = \sum_n A_n \sin \frac{n\pi x}{l} \tag{51}$$
$$\psi(x) = \sum_n \frac{n\pi c}{l} B_n \sin \frac{n\pi x}{l} \tag{52}$$

2.2 Neumann condition

2.2.1 Diffusion equation

$$u(x,t) = \frac{A_0}{2} + \sum_n A_n \mathrm{e}^{-(n\pi/l)^2 kt} \cos \frac{n\pi x}{l} \tag{53}$$

where

$$\phi(x) = \frac{A_0}{2} + \sum_n A_n \cos \frac{n\pi x}{l} \tag{54}$$

2.2.2 Wave equation

$$u(x,t) = \frac{A_0}{2} + \frac{B_0 t}{2} + \sum_n \left(A_n \cos \frac{n\pi ct}{l} + B_n \sin \frac{n\pi ct}{l} \right) \cos \frac{n\pi x}{l} \tag{55}$$

where

$$\phi(x) = \frac{A_0}{2} + \sum_n A_n \cos \frac{n\pi x}{l} \tag{56}$$
$$\psi(x) = \frac{B_0}{2} + \sum_n \frac{n\pi c}{l} B_n \cos \frac{n\pi x}{l} \tag{57}$$

3 Fourier series

3.1 Sine series

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l} \tag{58}$$
$$A_n = \frac{2}{l} \int_0^l \phi(x) \sin \frac{n\pi x}{l} \mathrm{d}x \tag{59}$$

3.2 Cosine series

$$\phi(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{l} \tag{60}$$
$$A_n = \frac{2}{l} \int_0^l \phi(x) \cos \frac{n\pi x}{l} \mathrm{d}x \tag{61}$$

3.3 Full series

$$\phi(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi x}{l} + B_n \sin\frac{n\pi x}{l}\right) \tag{62}$$

$$A_n = \frac{1}{l}\int_{-l}^{l}\phi(x)\cos\frac{n\pi x}{l}\mathrm{d}x \tag{63}$$

$$B_n = \frac{1}{l}\int_{-l}^{l}\phi(x)\sin\frac{n\pi x}{l}\mathrm{d}x \tag{64}$$

3.4 Complex form

$$\phi(x) = \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{in\pi x/l} \tag{65}$$

$$c_n = \frac{1}{2l}\int_{-l}^{l}\phi(x)\mathrm{e}^{-in\pi x/l}\mathrm{d}x \tag{66}$$

4 Harmonic functions

$$\Delta_2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} \tag{67}$$

$$\Delta_3 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \tag{68}$$

4.1 Solution

$$0 = u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} \tag{69}$$

Let $u(r,\theta) = R(r)\Theta(\theta)$

$$\begin{aligned} 0 &= R''\Theta + r^{-1}R'\Theta + r^{-2}R\Theta'' \\ 0 &= \frac{r^2R'' + rR'}{R} + \frac{\Theta''}{\Theta} \\ \frac{r^2R''+rR'}{R} &= -\frac{\Theta''}{\Theta} = \lambda \end{aligned} \tag{70}$$

where λ is a constant. Thus

$$\Theta'' + \lambda\Theta = 0 \tag{71}$$

$$r^2R'' + rR' - \lambda R = 0 \tag{72}$$

with solutions

$$\Theta(\theta) = c_1\cos\theta\sqrt{\lambda} + c_2\sin\theta\sqrt{\lambda} \tag{73}$$

$$R(r) = c_3 r^{\sqrt{\lambda}} + c_4 r^{-\sqrt{\lambda}} \tag{74}$$

Since $\Theta(\theta+2\pi) = \Theta(\theta)$, $\lambda = n^2$

$$\Theta(\theta) = c_1\cos n\theta + c_2\sin n\theta \tag{75}$$

$$R(r) = c_3r^n + c_4r^{-n} \tag{76}$$

In the case of $n = 0$, there is a second independent solution to $R(r)$:

$$R(r) = c_3 + c_4\log r \tag{77}$$

4.2 Poisson's formula

$$u_{xx} + u_{yy} = 0 \tag{78}$$

$$u = h(\theta) \tag{79}$$

$$u = \frac{A_0}{2} + \sum_{n=1}^{\infty} r^n(A_n\cos n\theta + B_n\sin n\theta) \tag{80}$$

$$h(\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} a^n(A_n\cos n\theta + B_n\sin n\theta) \tag{81}$$

$$A_n = \frac{1}{\pi a^n}\int_0^{2\pi} h(\phi)\cos n\phi\,\mathrm{d}\phi \tag{82}$$

$$B_n = \frac{1}{\pi a^n}\int_0^{2\pi} h(\phi)\sin n\phi\,\mathrm{d}\phi \tag{83}$$

$$u(r,\theta) = \frac{a^2-r^2}{2\pi}\int_0^{2\pi}\frac{h(\phi)\mathrm{d}\phi}{a^2 - 2ar\cos(\theta-\phi) + r^2} \tag{84}$$

4.3 Wedge

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^{n\pi/\beta}\sin\frac{n\pi\theta}{\beta} \tag{85}$$

4.4 Annulus

$$u(r,\theta) = \frac{C_0 + D_0\log r}{2} + \sum_{n=1}^{\infty}\left((C_nr^n + D_nr^{-n})\cos n\theta + (A_nr^n + B_nr^{\cdots}\right. \tag{86}$$

4.5 Exterior of circle

$$u(r,\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} r^{-n}(A_n\cos n\theta + B_n\sin n\theta) \tag{87}$$

$$= \frac{r^2-a^2}{2\pi}\int_0^{2\pi}\frac{h(\phi)\mathrm{d}\phi}{a^2 - 2ar\cos(\theta-\phi) + r^2} \tag{88}$$

5 Green's identities and Green's functions

5.1 Green's first identity

$$\nabla\cdot(v\nabla u) = \nabla v\cdot\nabla u + v\Delta u \tag{89}$$

$$\iint_{\partial D} v\frac{\partial u}{\partial n}\mathrm{d}S = \iiint_D \nabla v\cdot\nabla u\mathrm{d}V + \iiint_D v\Delta u\mathrm{d}V \tag{90}$$

Try letting $v = 1$, $v = u$, etc. to solve problems.

5.1.1 Mean value property

$$
\begin{aligned}
0 &= \iiint_B \Delta u \mathrm{d}V \\
&= \iint_{\partial B} \nabla u \cdot \hat{r} \mathrm{d}S \\
&= \iint_{\partial B} \frac{\partial u}{\partial r} \mathrm{d}S \\
&= \frac{\partial}{\partial r} \iint_{\partial B} u \mathrm{d}S
\end{aligned} \tag{91}
$$

Therefore the integral is independent of r. Therefore

$$
\begin{aligned}
\iint_{\partial B} u \mathrm{d}S &= \iint_{\partial B} u(0) \mathrm{d}S \\
&= u(0) \iint_{\partial B} \mathrm{d}S
\end{aligned} \tag{92}
$$

Therefore

$$
\begin{aligned}
u(0) &= \left(\iint_{\partial B} \mathrm{d}S \right)^{-1} \iint_{\partial B} u \mathrm{d}S \\
&= \frac{1}{\text{area}} \iint_{\partial B} u \mathrm{d}S
\end{aligned} \tag{93}
$$

5.1.2 Uniqueness of Robin problem

$$\Delta u_1 = \Delta u_2 = f$$

$$\frac{\partial u_1}{\partial n} + a u_1 = \frac{\partial u_2}{\partial n} + a u_2 = h$$

Let $u = u_1 - u_2$

$$
\begin{aligned}
\Delta u &= \Delta(u_1 - u_2) \\
&= \Delta u_1 - \Delta u_2 \\
&= f - f \\
&= 0
\end{aligned} \tag{94}
$$

$$
\begin{aligned}
\frac{\partial u}{\partial n} + au &= \frac{\partial (u_1 - u_2)}{\partial n} + a(u_1 - u_2) \\
&= \left(\frac{\partial u_1}{\partial n} + a u_1 \right) - \left(\frac{\partial u_2}{\partial n} + a u_2 \right) \\
&= h - h \\
&= 0
\end{aligned} \tag{95}
$$

From Green's first identity

$$\iint v \nabla u = \iiint (\nabla v \cdot \nabla u + v \Delta u) \tag{96}$$

Let $v = u$

$$
\begin{aligned}
\iint u \nabla u &= \iiint (\nabla u \cdot \nabla u + u \Delta u) \\
\iint u \frac{\partial u}{\partial n} &= \iiint |\nabla u|^2 \\
\iint u(-au) &= \\
-a \iint u^2 &=
\end{aligned} \tag{97}
$$

Since $a > 0$, the LHS ≤ 0 and the RHS ≥ 0. Therefore LHS = RHS = 0.

$$0 = -a \iint u^2 \tag{98}$$

$$0 = \iiint |\nabla u|^2 \tag{99}$$

On the boundary, we have

$$
\begin{aligned}
u^2 &\geq 0 \\
0 &= \iint u^2 \\
\therefore u^2 &= 0 \\
\therefore u &= 0
\end{aligned}
$$

In the interior we have

$$
\begin{aligned}
|\nabla u|^2 &\geq 0 \\
0 &= \iint |\nabla u|^2 \\
\therefore |\nabla u|^2 &= 0 \\
\therefore \nabla u &= 0 \\
\therefore u &= \text{constant}
\end{aligned}
$$

Since $u =$ constant in D and $u = 0$ on ∂D, $u = 0$ everywhere in D as well. $\therefore u_1 = u_2$ everywhere in D.

A similar technique can be used to prove the uniqueness of the Dirichlet problem, and the Neumann problem (up to an additive constant).

5.1.3 Maximum principle

A nonconstant harmonic function in D does not assume its maximum value inside D but only on ∂D. Also

$$\frac{\partial u}{\partial n} > 0$$

at a maximum point (Hopf maximum principle).

5.1.4 Dirichlet principle

$$E(w) = \frac{1}{2} \int_D |\nabla w|^2 \tag{100}$$

$$= \frac{1}{2} \int_D |\nabla (u - v)|^2 \tag{101}$$

$$= \frac{1}{2} \int_D |\nabla u - \nabla v|^2 \tag{102}$$

$$= \frac{1}{2} \int_D (\nabla u - \nabla v) \cdot (\nabla u - \nabla v) \tag{103}$$

$$= \frac{1}{2} \int_D (|\nabla u|^2 + |\nabla v|^2 - 2 \nabla u \cdot \nabla v \tag{104}$$

$$= E(u) + E(v) - \int_D \nabla u \cdot \nabla v \tag{105}$$

$$= E(u) + E(v) + \int_D v \nabla^2 u - \int_{\partial D} v \frac{\partial u}{\partial n} \tag{106}$$

$$= E(u) + E(v) + \int_D v \nabla^2 u \tag{107}$$

$$= E(u) + E(v) \tag{108}$$

But $E(v) \geq 0$. Therefore $E(w) \geq E(u) \Rightarrow E(u) \leq E(w)$ for any w provided u is harmonic.

5.1.5 Uniqueness of diffusion with Dirichlet conditions

$$(\partial_t - k\Delta)u_1 = (\partial_t - k\Delta)u_2 = 0$$

$$u_1|_{t=0} = u_2|_{t=0} = g$$

$$u_1|_{\partial D} = u_2|_{\partial D} = h$$

Let $u = u_1 - u_2$

$$\begin{aligned} u|_{t=0} &= (u_1 - u_2)|_{t=0} \\ &= u_1|_{t=0} - u_2|_{t=0} \\ &= g - g \\ &= 0 \end{aligned} \tag{109}$$

$$\begin{aligned} u|_{\partial D} &= (u_1 - u_2)|_{\partial D} \\ &= u_1|_{\partial D} - u_2|_{\partial D} \\ &= h - h \\ &= 0 \end{aligned} \tag{110}$$

$$\begin{aligned} (\partial_t - k\Delta)u &= (\partial_t - k\Delta)(u_1 - u_2) \\ &= (\partial_t - k\Delta)u_1 - (\partial_t - k\Delta)u_2 \\ &= 0 - 0 \\ &= 0 \end{aligned} \tag{111}$$

$$E(u) = \frac{1}{2}\int_D u^2 \tag{112}$$

$$E(u)|_{t=0} = \frac{1}{2}\int_D u^2|_{t=0} = \frac{1}{2}\int_D 0 = 0 \tag{113}$$

$$\begin{aligned} \frac{\partial E(u)}{\partial t} &= \frac{1}{2}\int_D \frac{\partial u^2}{\partial t} \\ &= \frac{1}{2}\int_D \frac{\partial u^2}{\partial u}\frac{\partial u}{\partial t} \\ &= \int_D u\frac{\partial u}{\partial t} \\ &= \int_D u\Delta u \\ &= \int_{\partial D} u\frac{\partial u}{\partial n} - \int_D \nabla u \cdot \nabla u \\ &= -\int_D |\nabla u|^2 \\ &\leq 0 \end{aligned} \tag{114}$$

$E(u) = 0$ for $t = 0$ and $\partial_t E(u) \leq 0$. Therefore $E(u) \leq 0$ for all $t > 0$. But $E(u) \geq 0$! Thus it must be the case that $E(u) = 0$ for all $t > 0$.

$$\frac{1}{2}\int_D u^2 = 0 \tag{115}$$

But $u^2 \geq 0$. Thus $u^2 = 0$ and $u = 0$. Thus $u_1 = u_2$.

5.1.6 Dirichlet principle for Neumann condition

$$E(w) = \frac{1}{2}\int_D |\nabla w|^2 - \int_{\partial D} hw \tag{116}$$

$$\Delta u = 0 \text{ on } D \tag{117}$$

$$\frac{\partial u}{\partial n} = h \text{ on } \partial D \tag{118}$$

$$\begin{aligned} E(w) &= E(u - v) \\ &= \frac{1}{2}\int_D |\nabla(u - v)|^2 - \int_{\partial D} h(u - v) \\ &= \frac{1}{2}\int_D |\nabla u - \nabla v|^2 - \int_{\partial D} (hu - hv) \\ &= \frac{1}{2}\int_D (|\nabla u|^2 + |\nabla v|^2 - 2\nabla u \cdot \nabla v) - \int_{\partial D} hu + \int_{\partial D} hv \\ &= \frac{1}{2}\int_D |\nabla u|^2 - \int_{\partial D} hu + \frac{1}{2}\int_D |\nabla v|^2 + \int_{\partial D} hv - \int_D \nabla u \cdot \nabla v \\ &= E(u) + \frac{1}{2}\int_D |\nabla v|^2 - \int_D \nabla u \cdot \nabla v + \int_{\partial D} hv \end{aligned} \tag{119}$$

Recalling Green's first identity

$$\int_D \nabla v \cdot \nabla u = \int_{\partial D} v\nabla u - \int_D v\Delta u \tag{120}$$

$$\begin{aligned} E(w) &= E(u) + \frac{1}{2}\int_D |\nabla v|^2 + \int_D v\Delta u - \int_{\partial D} v\nabla u + \int_{\partial D} hv \\ &= E(u) + \frac{1}{2}\int_D |\nabla v|^2 - \int_{\partial D} vh + \int_{\partial D} hv \\ &= E(u) + \frac{1}{2}\int_D |\nabla v|^2 \end{aligned} \tag{121}$$

Since $\frac{1}{2}\int_D |\nabla v|^2 \geq 0$, $E(w) \geq E(u)$ and $E(u) \leq E(w)$.

5.1.7 Rayleigh-Ritz approximation

Let

$$w = w_0 + \sum_{i=1}^{n} c_i w_i \tag{122}$$

Then

$$
\begin{aligned}
E(w) &= \frac{1}{2}\int_D |\nabla w|^2 \\
&= \frac{1}{2}\int_D \left|\nabla w_0 + \nabla \sum_{i=1}^{n} c_i w_i\right|^2 \\
&= \frac{1}{2}\int_D \left|\nabla w_0 + \sum_{i=1}^{n} c_i \nabla w_i\right|^2 \\
&= \frac{1}{2}\int_D \nabla w_0 \cdot \left(\nabla w_0 + \sum_{i=1}^{n} c_i \nabla w_i\right) + \sum_{i=1}^{n} c_i \nabla w_i \cdot \left(\nabla w_0 + \sum_{i=1}^{n} c_i \nabla w_i\right) \\
&= \frac{1}{2}\int_D \nabla w_0 \cdot w_0 + 2\nabla w_0 \cdot \sum_{i=1}^{n} c_i \nabla w_i + \sum_{i=1}^{n} c_i \nabla w_i \cdot \sum_{i=1}^{n} c_i \nabla w_i \\
&= \frac{1}{2}\int_D |\nabla w_0|^2 + \int_D \nabla w_0 \cdot \sum_{i=1}^{n} c_i \nabla w_i + \frac{1}{2}\int_D \left|\sum_{i=1}^{n} c_i \nabla w_i\right|^2 \\
&= \frac{1}{2}\int_D |\nabla w_0|^2 + \int_D \sum_{i=1}^{n} \nabla w_0 \cdot c_i \nabla w_i + \frac{1}{2}\int_D \sum_{i=1}^{n}\sum_{j=1}^{n} (c_i \nabla w_i)\cdot(c_j \nabla w_j) \\
&= \frac{1}{2}\int_D |\nabla w_0|^2 + \sum_{i=1}^{n} c_i \int_D \nabla w_0 \cdot \nabla w_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j \int_D \nabla w_i \cdot \nabla w_j
\end{aligned}
\tag{123}
$$

For the coefficients to minimize the energy we have

$$
\begin{aligned}
0 &= \frac{\partial E(w)}{\partial c_k} \\
&= \frac{1}{2}\frac{\partial}{\partial c_k}\int_D |\nabla w_0|^2 + \frac{\partial}{\partial c_k}\sum_{i=1}^{n} c_i \int_D \nabla w_0 \cdot \nabla w_i + \frac{1}{2}\frac{\partial}{\partial c_k}\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j \int_D \nabla w_i \cdot \nabla w_j \\
&= \sum_{i=1}^{n} \frac{\partial c_i}{\partial c_k}\int_D \nabla w_0 \cdot \nabla w_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial c_i c_j}{\partial c_k}\int_D \nabla w_i \cdot \nabla w_j \\
&= \sum_{i=1}^{n} \delta_{ik}\int_D \nabla w_0 \cdot \nabla w_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial c_i}{\partial c_k} c_j + c_i \frac{\partial c_j}{\partial c_k}\right)\int_D \nabla w_i \cdot \nabla w_j \\
&= \int_D \nabla w_0 \cdot \nabla w_k + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(\delta_{ik} c_j + c_i \delta_{jk})\int_D \nabla w_i \cdot \nabla w_j \\
&= (\nabla w_0, \nabla w_k) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(\delta_{ik} c_j + c_i \delta_{jk})(\nabla w_i, \nabla w_j) \\
&= (\nabla w_0, \nabla w_k) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\delta_{ik} c_j (\nabla w_i, \nabla w_j) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} c_i \delta_{jk}(\nabla w_i, \nabla w_j) \\
&= (\nabla w_0, \nabla w_k) + \frac{1}{2}\sum_{j=1}^{n} c_j (\nabla w_k, \nabla w_j) + \frac{1}{2}\sum_{i=1}^{n} c_i (\nabla w_i, \nabla w_k) \\
&= (\nabla w_0, \nabla w_k) + \sum_{j=1}^{n} c_j (\nabla w_k, \nabla w_j)
\end{aligned}
\tag{124}
$$

Therefore

$$0 = \frac{\partial E(w)}{\partial c_j} = (\nabla w_0, \nabla w_j) + \sum_{k=1}^{n} c_k (\nabla w_j, \nabla w_k) \tag{125}$$

or

$$\sum_{k=1}^{n} c_k (\nabla w_j, \nabla w_k) = -(\nabla w_0, \nabla w_j) \tag{126}$$

5.1.8 Variational principle behind Robin condition

$$
\begin{aligned}
\Delta u &= f & (127)\\
0 &= \nu \cdot \nabla u + au & (128)\\
&= \frac{\partial u}{\partial n} + au & (129)
\end{aligned}
$$

$$0 = \frac{\partial u}{\partial n} v + auv \tag{130}$$

$$
\begin{aligned}
0 &= \int_{\partial D} 0 & (131)\\
&= \int_{\partial D} 0v & (132)\\
&= \int_{\partial D}\left(\frac{\partial u}{\partial n} + au\right) v & (133)\\
&= \int_{\partial D}\frac{\partial u}{\partial n} v + \int_{\partial D} auv & (134)\\
&= \int_{\partial D}\frac{\partial u}{\partial n} v + a\int_{\partial D} uv & (135)\\
&= \int_D \nabla u \cdot \nabla v + \int_D v\Delta u + a\int_{\partial D} uv & (136)\\
&= \int_D \nabla u \cdot \nabla v + \int_D vf + a\int_{\partial D} uv & (137)
\end{aligned}
$$

$$
\begin{aligned}
-\int_D vf &= \int_D \nabla u \cdot \nabla v + a\int_{\partial D} uv & (138)\\
&= \int_D \nabla u \cdot \nabla v + a\int_D \nabla \cdot uv & (139)\\
&= \int_D (\nabla u \cdot \nabla v + a\nabla \cdot uv) & (140)
\end{aligned}
$$

Therefore

$$L(v) = A(u, v) \tag{141}$$

where

$$
\begin{aligned}
L(v) &= -\int_D vf & (142)\\
A(u, v) &= \int_D \nabla u \cdot \nabla v + a\int_{\partial D} uv & (143)
\end{aligned}
$$

The variational (weak formulation) problem consists of finding a function u such that $L(v) = A(u, v)$ for all functions v.

5.2 Green's second identity

$$\nabla \cdot (u\nabla v - v\nabla u) = u\nabla^2 v - v\nabla^2 u \tag{144}$$

$$\int_D (u\Delta v - v\Delta u) = \int_{\partial D}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) \tag{145}$$

5.2.1 Representation formula

Letting

$$v(x) = -\frac{1}{4\pi|x-x_0|} \tag{146}$$

we obtain

$$u(x_0) = \frac{1}{4\pi}\int_{\partial D}\left(-u(x)\frac{\partial}{\partial n}\frac{1}{|x-x_0|} + \frac{1}{|x-x_0|}\frac{\partial u(x)}{\partial n}\right) \tag{147}$$

where $\Delta u = 0$ and $\Delta v = 0$.

In 2 dimensions, $v(x) = \log|x-x_0|$

$$u(x_0) = \frac{1}{2\pi}\int_{\partial D}\left(u(x)\frac{\partial}{\partial n}\log|x-x_0| - \frac{\partial u(x)}{\partial n}\log|x-x_0|\right) \tag{148}$$

5.3 Green's functions

A Green's function for Δ in the domain D satisfies the following: $\Delta G(x) = 0$ in D except at $x = x_0$, $G(x) = 0$ in ∂D, and $G(x) + \frac{1}{4\pi|x-x_0|}$ is harmonic at x_0 and has continuous second derivatives everywhere.

The solution to the Dirichlet problem if $G(x, x_0)$ is the Green's function is

$$u(x_0) = \int_{\partial D} u(x)\frac{\partial G(x,x_0)}{\partial n} + \int_D f(x)G(x,x_0) \tag{149}$$

where $\Delta u = f$ in D and $u = h$ in ∂D.

6 Waves in space

6.1 Conservation of energy

$$E = \frac{1}{2}\int_D (u_t^2 + c^2|\nabla u|^2) \tag{150}$$

where the first term is the kinetic energy and the second term is the potential energy.

6.2 Causality principle

The initial data at a spatial point can influence the solution only in the solid (future) light cone emanating from that point.

6.3 Wave equation solution in 3+1 dimensions

Let $\Box$ be the spacelike d'Alembertian and

$$\Box u(x,t) = f(x,t) \tag{151}$$
$$u(x,0) = \phi(x) \tag{152}$$
$$u_t(x,0) = \psi(x) \tag{153}$$

Then

$$\begin{aligned} u(\xi,\tau) &= \frac{\partial}{\partial\tau}\frac{1}{4\pi\tau}\int_{\partial B_\tau(\xi)}\phi(x)\mathrm{d}x \\ &\quad + \frac{1}{4\pi\tau}\int_{\partial B_\tau(\xi)}\psi(x)\mathrm{d}x \\ &\quad - \int_0^\tau\left(\frac{1}{4\pi(\tau-t)}\int_{\partial B_{\tau-t}(\xi)} f(x,t)\mathrm{d}x\right)\mathrm{d}t \end{aligned} \tag{154}$$

for all $\xi \in \mathbb{R}^3$ and $\tau > 0$.

6.4 Huygens Principle

$u(\xi,\tau)$ is influenced only by the values of ϕ and ψ near the sphere $\partial B_\tau(\xi)$ and by the values of f along the backwards light cone.

7 Poisson formula in higher dimensions

$$u(x) = \frac{1-|x|^2}{\sigma_{n-1}}\int_{\partial B_1(0)}|x-y|^{-n}u(y)\,\mathrm{d}y \tag{155}$$

The Newtonian potential for the Laplacian operator is

$$\Gamma(x) = \begin{cases} \frac{1}{2\pi}\log|x| & n = 2 \\ \frac{|x|^{2-n}}{(2-n)\sigma_{n-1}} & n \geq 3 \end{cases} \tag{156}$$

The Green's function for the Dirichlet problem on the unit ball is

$$\begin{aligned} G(x,y) &= \Gamma(x-y) - |x|^{2-n}\Gamma\left(\frac{x}{|x|^2} - y\right) \\ &= \begin{cases} \frac{\log|x-y| - \log\left|\frac{x}{|x|} - |x|y\right|}{2\pi} & n = 2 \\ \frac{|x-y|^{2-n} - \left|\frac{x}{|x|} - |x|y\right|^{2-n}}{(2-n)\sigma_{n-1}} & n \geq 3 \end{cases} \end{aligned} \tag{157}$$

$$\begin{aligned} u(x) &= \int_{\partial B_1(0)}\langle\nabla_y G(x,y), \nu(y)\rangle u(y)\mathrm{d}y \\ &= \int_{\partial B_1(0)}\langle\nabla_y G(x,y), y\rangle u(y)\mathrm{d}y \\ &= -\int_{\partial B_1(0)}\frac{1}{\sigma_{n-1}}\left(|x-y|^{-n}\langle x-y,y\rangle - |\frac{x}{|x|} - |x|y|^{-n}\langle\frac{x}{|x|} - |x| \right. \\ &= -\int_{\partial B_1(0)}\frac{1}{\sigma_{n-1}}(|x-y|^{-n}\langle x-y,y\rangle - |x-y|^{-n}\langle x-|x|^2y, y\rangle)u \\ &= \int_{\partial B_1(0)}\frac{1-|x|^2}{\sigma_{n-1}}|x-y|^{-n}u(y)\mathrm{d}y \\ &= \frac{1-|x|^2}{\sigma_{n-1}}\int_{\partial B_1(0)}|x-y|^{-n}u(y)\mathrm{d}y \end{aligned} \tag{158}$$

7.1 Perron technique for the Dirichlet problem

u is a subharmonic function if $\Delta u \geq 0$. Also, the following statements are equivalent:

$$u(x_0) \le \frac{1}{\text{volume}(B_r(x_0))} \int_{B_r(x_0)} u \tag{159}$$

$$u(x_0) \le \frac{1}{\text{area}(\partial B_r(x_0))} \int_{\partial B_r(x_0)} u \tag{160}$$

8 Harnack's inequality

The solution formula for the Dirichlet problem on the unit ball is

$$u(x) = \frac{1-|x|^2}{\sigma_{n-1}} \int_{\partial B_1} \frac{1}{|x-y|^n} u(y)\mathrm{d}y \tag{161}$$

By the triangle inequality, we have

$$||x|-|y|| \le |x-y| \le ||x|+|y|| \tag{162}$$

Since $y \in \partial_{B_1}$, $|y| = 1$ and

$$||x|-1| \le |x-y| \le ||x|+1| \tag{163}$$

Since $|x| \le 1$, $|x|-1 \le 0$ and $||x|-1| = 1-|x|$. Hence

$$1-|x| \le |x-y| \le 1+|x| \tag{164}$$

Taking the reciprocals and reversing the order of the inequalities yields

$$\frac{1}{1+|x|} \le \frac{1}{|x-y|} \le \frac{1}{1-|x|} \tag{165}$$

Hence

$$\frac{1}{(1+|x|)^n} \le \frac{1}{|x-y|^n} \le \frac{1}{(1-|x|)^n} \tag{166}$$

and, since $1-|x|^2 > 0$,

$$\frac{1-|x|^2}{(1+|x|)^n} \le \frac{1-|x|^2}{|x-y|^n} \le \frac{1-|x|^2}{(1-|x|)^n} \tag{167}$$

Factoring $1-|x|^2$ into $(1-|x|)(1+|x|)$ yields

$$\frac{1-|x|}{(1+|x|)^{n-1}} \le \frac{1-|x|^2}{|x-y|^n} \le \frac{1+|x|}{(1-|x|)^{n-1}} \tag{168}$$

Therefore, since $u(y) \ge 0$,

$$\begin{aligned} \frac{1}{\sigma_{n-1}} \int_{\partial B_1} \frac{1-|x|}{(1+|x|)^{n-1}} u(y)\mathrm{d}y &\le \frac{1}{\sigma_{n-1}} \int_{\partial B_1} \frac{1-|x|^2}{|x-y|^n} u(y)\mathrm{d}y \\ &\le \frac{1}{\sigma_{n-1}} \int_{\partial B_1} \frac{1+|x|}{(1-|x|)^{n-1}} u(y)\mathrm{d}y \end{aligned} \tag{169}$$

Moving the x terms outside the integrals and substituting $u(x)$ yields

$$\frac{1}{\sigma_{n-1}} \frac{1-|x|}{(1+|x|)^{n-1}} \int_{\partial B_1} u(y)\mathrm{d}y \le u(x) \le \frac{1}{\sigma_{n-1}} \frac{1+|x|}{(1-|x|)^{n-1}} \int_{\partial B_1} u(y)\mathrm{d}y \tag{170}$$

The average of a harmonic function over a sphere is equal to its value at the center of the sphere:

$$\frac{1}{\sigma_{n-1}} \int_{B_1} u(y)\mathrm{d}y = u(0) \tag{171}$$

Therefore

$$\frac{1-|x|}{(1+|x|)^{n-1}} u(0) \le u(x) \le \frac{1+|x|}{(1-|x|)^{n-1}} u(0) \tag{172}$$