Relativistic field equations

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1 Klein-Gordon equation

The Klein-Gordon equation is the relativistic field equation for spin-0 particles. It can be derived from the relativistic energy-momentum relation

$$E^2 = (pc)^2 + (mc^2)^2 \tag{1}$$

by quantization, where the energy and momentum operators are

$$\begin{aligned} \hat{E} &= i\hbar\partial_t \\ \hat{p} &= -i\hbar\nabla \end{aligned} \tag{2}$$

Quantization yields

$$\begin{aligned} \hat{E}^2\psi &= (\hat{p}c)^2\psi + (mc^2)^2\psi \\ (i\hbar\partial_t)^2\psi &= (-i\hbar\nabla c)^2\psi + (mc^2)^2\psi \\ i^2\hbar^2\partial_t^2\psi &= i^2\hbar^2\nabla^2c^2\psi + m^2c^4\psi \\ -\hbar^2\partial_t^2\psi &= -\hbar^2\nabla^2c^2\psi + m^2c^4\psi \\ 0 &= \hbar^2\partial_t^2\psi - \hbar^2\nabla^2c^2\psi + m^2c^4\psi \\ 0 &= \frac{1}{c^2}\partial_t^2\psi - \nabla^2\psi + \frac{m^2c^2}{\hbar^2}\psi \\ 0 &= \Box^2\psi + \frac{m^2c^2}{\hbar^2}\psi \\ &= \left(\Box^2 + \frac{m^2c^2}{\hbar^2}\right)\psi \\ &= \left(\Box^2 + \mu^2\right)\psi \end{aligned} \tag{3}$$

where $\Box^2 = \partial_\mu\partial^\mu$ is the d'Alembertian. Alternatively, the equation can be derived from the four-momentum P_μ and its associated operator $i\hbar\partial_\mu$.

$$
\begin{aligned}
\hat{P}_\mu \hat{P}^\mu \psi &= P_\mu P^\mu \psi \\
(i\hbar\partial_\mu)(i\hbar\partial^\mu)\psi &= (mc)^2\psi \\
i^2\hbar^2\partial_\mu\partial^\mu\psi &= (mc)^2\psi \\
\partial_\mu\partial^\mu\psi &= \frac{(mc)^2}{i^2\hbar^2}\psi \\
\Box^2\psi &= -\left(\frac{mc}{\hbar}\right)^2\psi \\
\Box^2\psi + \left(\frac{mc}{\hbar}\right)^2\psi &= 0 \\
\left(\Box^2 + \left(\frac{mc}{\hbar}\right)^2\right)\psi &= 0 \\
(\Box^2 + \mu^2)\psi &= 0
\end{aligned} \tag{4}
$$

2 Dirac equation

The Dirac equation is the relativistic field equation for spin-1/2 particles. It can be derived as follows

$$
\begin{aligned}
\hat{P}\psi &= P\psi \\
\hat{P}_\mu\psi &= P_\mu\psi \\
i\hbar\partial_\mu\psi &= P_\mu\psi \\
\gamma^\mu i\hbar\partial_\mu\psi &= \gamma^\mu P_\mu\psi \\
i\hbar\gamma^\mu\partial_\mu\psi &= \gamma^\mu P_\mu\psi \\
i\hbar\gamma^\mu\partial_\mu\psi &= \sqrt{P^\mu P_\mu}\psi \\
i\hbar\gamma^\mu\partial_\mu\psi &= \sqrt{m^2c^2}\psi \\
i\hbar\gamma^\mu\partial_\mu\psi &= mc\psi \\
i\hbar\partial\!\!\!/\psi &= mc\psi \\
(mc - i\hbar\partial\!\!\!/)\psi &= 0
\end{aligned} \tag{5}
$$

where γ^μ are gamma matrices, also known as Dirac matrices. They generate the Clifford algebra $C\ell_{1,3}(R)$ and satisfy the anticommutation relation

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu} \tag{6}$$

Here the Feynman slash notation $a\!\!\!/ = \gamma^\mu a_\mu$ has been used. Since

$$a\!\!\!/ a\!\!\!/ = a^2 = a_\mu a^\mu \tag{7}$$

the gamma matrices provide us, in a sense, with 'square roots'

$$\sqrt{a_\mu a^\mu} = \not{a} = \gamma^\mu a_\mu \tag{8}$$

Notice also that

$$\begin{aligned} |(mc - i\hbar\not{\partial})|^2 &= (mc - i\hbar\not{\partial})(mc - i\hbar\not{\partial})^* \\ &= (mc - i\hbar\not{\partial})(mc + i\hbar\not{\partial}) \\ &= (mc)^2 - (i\hbar\not{\partial})^2 \\ &= m^2c^2 - i^2\hbar^2\not{\partial}^2 \\ &= m^2c^2 + \hbar^2\Box^2 \end{aligned} \tag{9}$$

Hence any component of any solution to the Dirac equation is also a solution to the Klein-Gordon equation.

3 Relativistic electrodynamics

The electromagnetic field is described by the electromagnetic four-potential A^μ. This four-vector contains the electric potential in its temporal component and the magnetic potential in its spatial component.

The electromagnetic field tensor is defined in terms of the potential as

$$F = \mathrm{d}A \tag{10}$$

where d is the exterior derivative. Equivalently,

$$F_{\mu\nu} = \partial_{[\mu}A_{\nu]} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{11}$$

The Lorenz gauge condition is a Lorentz invariant condition where

$$\Box \cdot A = \partial_\mu A^\mu = 0 \tag{12}$$

Maxwell's equations can then be expressed in a single elegant equation:

$$\Box^2 A^\mu = \partial_\nu \partial^\nu A^\mu = J^\mu \tag{13}$$

where J^μ is the four-current, which contains the charge density and the conventional current density in its temporal and spatial components, respectively. The continuity equation for the four-current, which expresses conservation in a local (differential) form, is simply

$$\Box \cdot J = \partial_\mu J^\mu = 0 \tag{14}$$

4 Relativistic heat conduction

The non-relativistic heat equation is

$$(\partial_t - \alpha\nabla^2)\theta = 0 \tag{15}$$

where ∇^2 is the Laplacian:

$$\nabla^2 = \partial_i\partial^i \tag{16}$$

θ is temperature, t is time, and α is thermal diffusivity. The heat flux density k, which is the rate of heat transfer per unit area per unit time, is

$$q = -k\nabla\theta \tag{17}$$

where k is thermal conductivity. The relativistic (Lorentz invariant) version of the heat equation is

$$(\partial_t - \alpha\Box^2)\theta = 0 \tag{18}$$

where $\Box^2$ is the (spacelike) d'Alembertian:

$$\Box^2 = -\frac{1}{c^2}\partial_t^2 + \nabla^2 \tag{19}$$

Similarly, the heat flux density is

$$q = -k\Box\theta \tag{20}$$

where $\Box$ is the four-gradient

$$\Box = \mathbf{e}_t\frac{1}{ic}\partial_t + \nabla \tag{21}$$

Hence we obtain the first law of thermodynamics

$$\frac{k}{\alpha}\partial_t\theta + \Box\cdot q = 0 \tag{22}$$

Similarly, we obtain the second law of thermodynamics

$$\Box\cdot\frac{q}{\theta} + \rho\partial_t s = \sigma \tag{23}$$

where ρ is density, s is specific entropy, and σ is entropy production. It's very interesting to see that the second law of thermodynamics emerges from applying Lorentz covariance to the classical heat equation, much like magnetism emerges from the combination of electrostatics and Lorentz covariance.

5 Relativistic fluid dynamics

The relativistic Euler equations are a covariant generalization of the Euler equations, which describe adiabatic and inviscid (non-viscuous) flow.

A ‘perfect fluid’ is a fluid that can be completely characterized by its rest frame mass density ρ; and isotropic pressure p. It has no shear stresses, viscosity, or heat conduction. The stress-energy tensor $T^{\mu\nu}$ of a perfect fluid is

$$T^{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) u^\mu u^\nu + p g^{\mu\nu} \tag{24}$$

where u is the four-velocity of the fluid, satisfying $u_\mu u^\mu = c^2$. If the perfect fluid is also pressureless, we obtain a *dust*:

$$T^{\mu\nu} = \rho u^\mu u^\nu \tag{25}$$

The equations of motion of the fluid can be recovered from the continuity equation (conservation law) of the stress-energy tensor:

$$\nabla_\mu T^{\mu\nu} = 0 \tag{26}$$

Substituting our $T^{\mu\nu}$ yields

$$\nabla_\mu \left(\rho + pc^{-2}\right) u^\mu u^\nu + \nabla_\mu p g^{\mu\nu} = 0 \tag{27}$$

Expanding the first term, and using the product rule on the second term,

$$\nabla_\mu \rho u^\mu u^\nu + \nabla_\mu p c^{-2} u^\mu u^\nu + \left(\nabla_\mu p\right) g^{\mu\nu} + p \nabla_\mu g^{\mu\nu} = 0 \tag{28}$$

Using the product rule once again on the first term yields

$$\left(\nabla_\mu \rho u^\mu\right) u^\nu + \rho u^\mu \nabla_\mu u^\nu + \nabla_\mu p c^{-2} u^\mu u^\nu + \left(\nabla_\mu p\right) g^{\mu\nu} + p \nabla_\mu g^{\mu\nu} = 0 \tag{29}$$

The continuity equation for the amount of fluid is

$$\nabla_\mu \rho u^\mu = 0 \tag{30}$$

Hence

$$\rho u^\mu \nabla_\mu u^\nu + \nabla_\mu p c^{-2} u^\mu u^\nu + \left(\nabla_\mu p\right) g^{\mu\nu} + p \nabla_\mu g^{\mu\nu} = 0 \tag{31}$$

The divergence of the metric tensor is zero

$$\nabla_\mu g^{\mu\nu} = 0 \tag{32}$$

Hence

$$\rho u^\mu \nabla_\mu u^\nu + \nabla_\mu p c^{-2} u^\mu u^\nu + \left(\nabla_\mu p\right) g^{\mu\nu} = 0 \tag{33}$$

Finally, using tensor contraction on the last term yields

$$\rho u^\mu \nabla_\mu u^\nu + \nabla_\mu p c^{-2} u^\mu u^\nu + \nabla^\nu p = 0 \tag{34}$$

We now turn to the Cauchy momentum equation in the non-relativistic Euler equation:

$$0 = \rho \left(\frac{\partial}{\partial t} + \vec{u} \cdot \vec{\nabla} \right) \vec{u} + \vec{\nabla} p \tag{35}$$

It can be derived from a conservation law for the momentum $\rho\vec{u}$: The generation of momentum is equal to the force density f, hence

$$f = \left(\frac{\partial}{\partial t} \rho \vec{u} + \vec{\nabla} \cdot \rho \vec{u}\vec{u} \right) \tag{36}$$

where $\vec{u}\vec{u}$ is a dyadic product. Applying the product rule yields

$$f = \vec{u} \left(\frac{\partial}{\partial t} \rho + \vec{\nabla} \cdot \rho \vec{u} \right) + \rho \frac{\partial \vec{u}}{\partial t} + \rho \vec{u} \cdot \vec{\nabla} \vec{u} \tag{37}$$

The second factor in the first term vanishes from the continuity equation for ρ, which expresses the conservation of mass:

$$\frac{\partial}{\partial t} \rho + \vec{\nabla} \cdot \rho \vec{u} = 0 \tag{38}$$

Hence

$$f = \rho \frac{\partial \vec{u}}{\partial t} + \rho \vec{u} \cdot \vec{\nabla} \vec{u} = \rho \left(\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u} \right) \tag{39}$$

Letting $f = -\nabla p$ we obtain the desired relation

$$0 = \rho \left(\frac{\partial}{\partial t} + \vec{u} \cdot \vec{\nabla} \right) \vec{u} + \vec{\nabla} p = \rho \left(c\nabla_0 + \vec{u} \cdot \vec{\nabla} \right) \vec{u} + \vec{\nabla} p \tag{40}$$

Using the non-relativistic approximation $\gamma \approx 1$ we obtain:

$$0 \approx \rho \left(\gamma c \nabla_0 + \gamma \vec{u} \cdot \vec{\nabla} \right) \gamma \vec{u} + \vec{\nabla} p = \rho u^\mu \nabla_\mu u^i + \nabla^i p \tag{41}$$

Compare this with the result obtained from the stress-energy tensor:

$$0 = \rho u^\mu \nabla_\mu u^\nu + \nabla_\mu p c^{-2} u^\mu u^\nu + \nabla^\nu p \tag{42}$$

On a tangential note, an interesting fact is that there exists a hydrodynamic interpretation of the Schrodinger equation called the Madelung equations. Can we similarly upgrade these equations to their covariant form and obtain a correct relativistic generalization of the Schrodinger equation?